informs
https://pubsonline.informs.org/journal/moor

MATHEMATICS OF OPERATIONS RESEARCH
*Articles in Advance*, pp. 1–24
ISSN 0364-765X (print), ISSN 1526-5471 (online)

# Value-Positivity for Matrix Games

**Krishnendu Chatterjee,[a] Miquel Oliu-Barton,[b] Raimundo Saona[a,*]**

[a] Institute of Science and Technology Austria, 3400 Klosterneuburg, Austria; [b] CEREMADE, CNRS, Université Paris Dauphine-PSL University, 75016 Paris, France

*Corresponding author

**Contact:** krishnendu.chatterjee@ist.ac.at, https://orcid.org/0000-0002-4561-241X (KC); miquel.oliu.barton@normalesup.org, https://orcid.org/0000-0003-1793-6350 (MO-B); raimundojulian.saonaurmeneta@ista.ac.at, https://orcid.org/0000-0001-5103-038X (RS)

**Received:** December 1, 2022
**Revised:** October 13, 2023; April 20, 2024
**Accepted:** August 5, 2024
**Published Online in Articles in Advance:** October 1, 2024

**MSC2020 Subject Classifications:** Primary: 91A05, 90C31, 90C05

https://doi.org/10.1287/moor.2022.0332



**Abstract.** Matrix games are the most basic model in game theory, and yet robustness with respect to small perturbations of the matrix entries is not fully understood. In this paper, we introduce value positivity and uniform value positivity, two properties that refine the notion of optimality in the context of polynomially perturbed matrix games. The first concept captures how the value depends on the perturbation parameter, and the second consists of the existence of a fixed strategy that guarantees the value of the unperturbed matrix game for every sufficiently small positive parameter. We provide polynomial-time algorithms to check whether a polynomially perturbed matrix game satisfies these properties. We further provide the functional form for a parameterized optimal strategy and the value function. Finally, we translate our results to linear programming and stochastic games, where value positivity is related to the existence of robust solutions.

**Funding:** This research was supported by Fondation CFM pour la Recherche, the H2020 European Research Council [Grant ERC-CoG-863818 (ForM-SMArt)], the Austrian Science Fund [Grant 10.55776/COE12], ANID Chile [Grant ACT210005], and Agence Nationale de la Recherche [Grant ANR-21-CE40-0020].

**Keywords:** stability • robustness • matrix games • linear programming

## 1. Introduction

Matrix games are the most basic model in game theory, where two opponents face each other over finitely many actions. The existence of the value and optimal strategies was established nearly 100 years ago (von Neumann [49]) and yet, surprisingly, robustness with respect to small perturbations of the matrix entries is not fully understood.

We consider *polynomial matrix games*, that is, matrix games where the entries are polynomial functions of some real parameter $\varepsilon \geq 0$. The matrix corresponding to $\varepsilon = 0$ is interpreted as the unperturbed game, as opposed to the other matrices, which are perturbed games. In a recent paper, Attia and Oliu-Barton [3] pointed out that polynomial matrix games arise naturally in the study of stochastic games, the first dynamic game introduced in the literature (Shapley [41], Solan and Vieille [43]).

In the context of polynomially perturbed matrix games, we introduce *value positivity* and *uniform value positivity*, two properties that refine the notion of optimality in matrix games. Whereas the first captures the dependency of optimal strategies with respect to perturbations, the second consists of the existence of a fixed strategy guaranteeing the value of the unperturbed matrix game for all sufficiently small positive parameters. Applied to linear programs (LPs), which are known to be equivalent to matrix games (Adler [1], Dantzig [14]), and stochastic games, these properties imply the existence of robust solutions and optimal strategies, respectively.

### Matrix Games

A matrix game is a game played between two opponents. It is represented by a matrix $M_0$, where rows (respectively, columns) correspond to possible actions for the row (respectively, column)-player, and the entry $(M_0)_{i,j}$ is the reward the column-player pays the row-player when the pair of actions $(i, j)$ is chosen. The two players choose actions simultaneously and independently, possibly randomizing their choices. Randomized strategies are called mixed strategies. The value of a matrix game, denoted val $M_0$, is the maximum amount the row-player can guarantee, that is, the amount one can obtain regardless of the column-player's strategy. By the min-max theorem (von Neumann [49]), it is also the minimum amount the column-player can guarantee, and both players have optimal mixed strategies. The value can be computed by solving a linear program, which can be solved in polynomial time (Khachiyan [27]).

### Polynomial Matrix Games

A polynomial matrix game M is described by $(K+1)$ matrices, $\mathrm{M}_0, \mathrm{M}_1, \mathrm{M}_2, \ldots, \mathrm{M}_K$. We associate to each $\varepsilon \geq 0$ the reward $\mathrm{M}(\varepsilon) = \mathrm{M}_0 + \mathrm{M}_1\varepsilon + \cdots + \mathrm{M}_K\varepsilon^K$. The parameter $\varepsilon$ represents the magnitude of a perturbation. The matrix game $M(0) = M_0$ is referred to as the unperturbed matrix game, whereas $\mathrm{M}_1$ is the first-order perturbation, $\mathrm{M}_2$ is the second-order perturbation, and so on.

### Value-Positivity Problems

We aim to understand how the value and the optimal strategies of a polynomial matrix game vary with the parameter. For every mixed strategy $\boldsymbol{p}$ of the row-player, we denote the amount that the row-player guarantees in the matrix game $M(\varepsilon)$ when playing the strategy $\boldsymbol{p}$ by $\mathrm{val}\,(\mathrm{M}(\varepsilon); \boldsymbol{p})$. We introduce value positivity to capture the existence of strategies that are robust to perturbations. More precisely, we consider two properties: (i) *value positivity* asks whether there exists $\varepsilon_0 > 0$ such that, for all $\varepsilon \in [0, \varepsilon_0]$ there is a strategy $\boldsymbol{p}_\varepsilon$ such that $\mathrm{val}\,(\mathrm{M}(\varepsilon); \boldsymbol{p}_\varepsilon) \geq \mathrm{val}\,(\mathrm{M}_0)$, and (ii) *uniform value positivity* asks whether there exists a fixed strategy $\boldsymbol{p}$ and $\varepsilon_0 > 0$ such that $\mathrm{val}\,(\mathrm{M}(\varepsilon); \boldsymbol{p}) \geq \mathrm{val}\,(\mathrm{M}_0)$ holds for all $\varepsilon \in [0, \varepsilon_0]$. Lastly, we consider the *functional form* problem, which consists of giving the analytical form of the function $\varepsilon \longmapsto \mathrm{val}\,\mathrm{M}(\varepsilon)$, and of an optimal strategy, in some right neighborhood of zero. Without loss of generality, we assume $\mathrm{val}\,\mathrm{M}_0 = 0$ throughout the paper, as the value operator translates constants; that is, for every matrix A and constant $c \in \mathbb{R}$, we have that $\mathrm{val}\,(\mathrm{A} + c\mathrm{U}) = \mathrm{val}\,(\mathrm{A}) + c$, where U is a matrix of ones of the same size as A. This justifies the terminology of value positivity and uniform value positivity.

### Relevance

Both value positivity and uniform value positivity capture the existence of *robust strategies*, where robustness is to be understood as guaranteeing the value of the unperturbed value for all sufficiently small perturbations. Uniform value positivity is a stronger notion, where the strategy is fixed, as opposed to value positivity where the strategy can depend on $\varepsilon$. Relying on the equivalence between matrix games and linear programming, we translate the notion of value positivity to perturbed linear programming. Further, we exploit the connection between stochastic games and polynomial matrix games from Attia and Oliu-Barton [3] to connect value positivity with lower bounds on the discounted and undiscounted values of a stochastic game and uniform value positivity with the existence of an optimal stationary strategy in undiscounted stochastic games.

### Linear Programming

Linear programming is an optimization problem where both the objective function and the (finitely many) constraints are linear (Dantzig [15], Kuhn and Tucker [28], Rockafellar [39]). An LP $(P)$ is represented by a matrix A and vectors $\boldsymbol{b}$ and $\boldsymbol{c}$; the objective is to maximize the function $\boldsymbol{x} \longmapsto \boldsymbol{c}^\top \boldsymbol{x}$ subject to the constraints $\mathrm{A}\boldsymbol{x} \leq \boldsymbol{b}$ and $\boldsymbol{x} \geq \boldsymbol{0}$. The value of the LP, denoted $\mathrm{val}\,(P)$, is $+\infty$ if the maximum is unbounded and $-\infty$ if constraints are infeasible, or it belongs to $\mathbb{R}$ otherwise. A vector is feasible in $(P)$ if it satisfies the constraints. A solution of $(P)$ is a feasible vector $\boldsymbol{x}^* \in \mathbb{R}^n$ such that $\mathrm{val}\,(P) = \boldsymbol{c}^\top \boldsymbol{x}^*$. By the strong duality theorem (Adler [1], Dantzig [15]), LPs have a related dual LP. Moreover, as Brooks and Reny [12] put it, LPs are equivalent to matrix games "both in the sense that a solution to any matrix game can be obtained by solving a suitably chosen LP problem and vice versa, as well as in the sense that their fundamental theorems, minimax and strong duality, each follow from the other."

### Perturbed LPs

Analogous to our approach for matrix games, we consider that $\mathrm{A}, \boldsymbol{b}, \boldsymbol{c}$ allow perturbations modeled by polynomials; that is, $\mathrm{A}(\varepsilon) = \mathrm{A}_0 + \mathrm{A}_1\varepsilon + \cdots + \mathrm{A}_k\varepsilon^k$, $\boldsymbol{b}(\varepsilon) = \boldsymbol{b}_0 + \boldsymbol{b}_1\varepsilon + \cdots + \boldsymbol{b}_k\varepsilon^k$, $\boldsymbol{c}(\varepsilon) = \boldsymbol{c}_0 + \boldsymbol{c}_1\varepsilon + \cdots + \boldsymbol{c}_k\varepsilon^k$. Denote $(P(\varepsilon))$ the LP given by the $\mathrm{A}(\varepsilon), \boldsymbol{b}(\varepsilon)$ and $\boldsymbol{c}(\varepsilon)$. We aim to understand how the value and the optimal strategies vary with the parameter, so we look into optimization-based interpretations of robustness. We consider how feasibility, optimal points, and the value function $\varepsilon \longmapsto \mathrm{val}\,(P(\varepsilon))$ change for sufficiently small positive parameters. Similar to the value-positivity problems, we consider the following robustness problems: (i) the *weak robustness* problem asks whether there exists a sufficiently small $\varepsilon_0 > 0$ such that, for all $\varepsilon \leq \varepsilon_0$, the corresponding LP is feasible and bounded, that is, if $\mathrm{val}\,(P(\varepsilon)) \in \mathbb{R}$; and (ii) the *strong robustness* problem asks whether there exists a fixed vector $\boldsymbol{x}^*$ and sufficiently small $\varepsilon_0 > 0$ such that, for all $\varepsilon \leq \varepsilon_0$, the vector $\boldsymbol{x}^*$ is a solution of $(P(\varepsilon))$. Also, we consider the *functional form* problem, which consists of computing the analytical form of the functions $\varepsilon \longmapsto \mathrm{val}\,(P(\varepsilon))$ and $\varepsilon \longmapsto x^*_\varepsilon$.

### Previous Results

The continuity and Lipschitz property of the value function of matrix games is well-known. Because the value function is not Fréchet differentiable, the stability analysis of matrix games is a classical problem usually investigated through directional derivatives. Mills [33] considered a linearly perturbed matrix game, that is, $\varepsilon \to \mathrm{M}_0 + \varepsilon \mathrm{M}_1$, also framed as a perturbation of $\mathrm{M}_0$ in the direction of $\mathrm{M}_1\varepsilon$. A characterization was obtained for the right derivative of this function, together with a polynomial-time algorithm, based on the reduction to an auxiliary LP of similar size. Value positivity and uniform value positivity were never addressed before.

### Our Contributions

Our main contributions are the following.

1. We show that value positivity and uniform value positivity problems differ and cannot be derived from Mills [33], even for linear perturbations, because the value can have a nonlinear dependency on the perturbation parameter (see Example 1).
2. We present polynomial-time algorithms to solve value positivity, uniform value positivity, and functional form problems for polynomial matrix games.
3. We apply this approach to perturbed linear programming to derive similar robustness results and show that weak (respectively, strong) robustness for perturbed LPs has a polynomial-time reduction to value positivity (respectively, uniform value positivity) for polynomial matrix games.

### Related Works

Perturbed matrix games go back to Mills [33], who characterized the right derivative of the value function. This result was extended to a broader class of games in Sorin [44], using an operator approach. Perturbed games can be seen as a set of parametrized games; the regularity of the value function and the set of $\varepsilon$-optimal strategies were studied in Tijs and Vrieze [47] for a broad class of games. The asymptotic behavior of perturbed games has been studied in Altman et al. [2].

More generally, perturbation theory is a broad field, and general perturbed linear programs have been studied (Avrachenkov et al. [7]). Previous research has focused on perturbations only to the objective function or the right-hand side of the constraints (Fiacco [18, 19], Gal [20], Gal et al. [21]), whereas research on general perturbation that affects the entries of the matrix is sparse (Jeroslow [24, 25]). LPs with general perturbations can exhibit very different behaviors; for example, feasibility can change for small perturbations, or the limit of optimal strategies as the perturbation vanishes is no longer feasible. Therefore, assumptions on the effect of the perturbation are usually taken. Checking whether these conditions are satisfied may take exponential time. In our case, we make the following structural assumption: both the primal and dual perturbed problems have uniformly bounded feasible regions. Under this class of perturbed LPs, our approach yields polynomial-time algorithms. For a general introduction to perturbation theory, see Avrachenkov et al. [7], who also present general computational results where the best algorithm so far takes exponential time in the worst case.

A particular class of perturbed LPs with efficient algorithms is described in Ruhe [40], studying parametric flows, which boils down to studying perturbed LPs where only the objective function depends on the perturbation, and therefore, efficient methods can be applied. Imprecise matrix games include interval matrix games that have been studied, for example, in Verma and Kumar [48], where an exponential time algorithm is given to compute the value and optimal strategies.

### Outline

First, we present the analysis for value-positivity problems: Section 2 introduces basic definitions, classical results, and our main contribution; Section 3 clarifies the relationship between the value-positive and uniform value-positive problems and their connection with Mills [33]; and Section 4 presents polynomial-time algorithms for value-positivity, uniform value-positivity, and functional form problems. Second, we present a reduction from robustness problems of perturbed LPs to value-positivity problems of polynomial matrix games in Section 5. Third, we provide new ways to analyze stochastic games using value positivity and uniform value positivity in Section 6. Finally, Section 7 discusses a couple of natural extensions of our results, and Section 8 presents the conclusion.

## 2. Preliminaries

In this section, we introduce basic definitions, classical results, and our main contribution.

### 2.1. Basic Definitions

For an integer $m \in \mathbb{N}$, we denote the set of integers from one to $m$ by $[m] := \{1, 2, \ldots, m\}$ and the set of probability distributions over $[m]$ by $\Delta[m]$. In the sequel, we consider matrices with rational entries. Let us start with basic definitions such as matrix games, linear programs, and some of their connections.

**Definition 1** (Matrix Games). A matrix game is given by a matrix M of size $m \times m$. Both players have the same strategy set $\Delta[m]$. Given $\boldsymbol{p} \in \Delta[m]$, denote $(\boldsymbol{p}^\top \mathrm{M})_j$, the reward the row-player obtains by using strategy $\boldsymbol{p}$ if the column-player plays action $j \in [m]$ in the matrix game M. The value of the matrix game M is

$$\operatorname{val} \mathrm{M} := \max_{p \in \Delta[m]} \min_{j \in [m]} (\boldsymbol{p}^\top \mathrm{M})_j.$$

An *optimal strategy* for the row-player is a strategy $\boldsymbol{p} \in \Delta[m]$ such that, for all $j \in [m]$, we have that $(\boldsymbol{p}^\top \mathrm{M})_j \geq \operatorname{val} \mathrm{M}$. Properties of the value for matrix games, as well as the continuity of the value function, go back to von Neumann [49].

**Definition 2** (Linear Program). An *LP* of size $m \times m$ is given by

$$(P) \begin{cases} \max\limits_{x} & \boldsymbol{c}^\top \boldsymbol{x} \\ s.t. & \mathrm{A}\boldsymbol{x} \leq \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0}, \end{cases}$$

where the matrix A has size $m \times m$. The value of the LP, denoted $\operatorname{val}(P)$, is $+\infty$ if the maximum is unbounded and $-\infty$ if constraints are infeasible, and it belongs to $\mathbb{R}$ otherwise. We say a vector $\boldsymbol{x} \in \mathbb{R}^n$ is feasible for $(P)$ if $\mathrm{A}\boldsymbol{x} \leq \boldsymbol{b}$ and $\boldsymbol{x} \geq \boldsymbol{0}$. An LP is feasible if it has a feasible vector. For a feasible LP with bounded value, a feasible vector $\boldsymbol{x}^* \in \mathbb{R}^n$ is a solution if $\operatorname{val}(P) = \boldsymbol{c}^\top \boldsymbol{x}^* \geq \boldsymbol{c}^\top \boldsymbol{x}$ for every feasible vector $\boldsymbol{x}$.

**LP for a Matrix Game.** Computing the value of a matrix game M can be done in polynomial time by solving a linear program such that $\operatorname{val}(P_{\mathrm{M}}) = \operatorname{val} \mathrm{M}$, given as follows.

$$(P_{\mathrm{M}}) \begin{cases} \max\limits_{\boldsymbol{p}, z} & z & \\ s.t. & (\boldsymbol{p}^\top \mathrm{M})_j \geq z, & \forall j \in [m] \cdot \\ & \boldsymbol{p} \in \Delta([m]) & \end{cases}$$

**Simplifications.** For our purposes, it is enough to consider matrices M and A and vectors $\boldsymbol{b}$ and $\boldsymbol{c}$ that have integer entries. This is without loss of generality by standard linear transformations. For example, for matrix games, the value function satisfies (i) for all positive scalars $\lambda$, we have $\operatorname{val}(\lambda \mathrm{M}) = \lambda \operatorname{val} \mathrm{M}$, and (ii) for all scalars $r$, we have $\operatorname{val}(r + \mathrm{M}) = r + \operatorname{val} \mathrm{M}$. Also, for LPs, inequalities can be multiplied by positive constants, and the objective function can be multiplied by positive scalars, as its value follows a similar relationship to the value of matrix games. Note that such linear transformations have no consequences in the complexity of algorithms, as the value of a matrix game with rational entries is the solution of an LP with rational entries and therefore has polynomial binary size. Finally, the fact that M and A are square is also without loss of generality, as, for matrix games, rows and columns can be duplicated, and for LPs, constraints can be duplicated, and slack variables can be introduced.

We now turn to perturbed matrix games and the value-positivity problems.

**Definition 3** (Polynomial Matrix Games). A *polynomial matrix game* $\mathrm{M}(\cdot)$ of degree $K$ and size $m \times m$ is a square matrix whose entries are polynomials of degree $K$ with integer coefficients; that is, for all $\varepsilon \geq 0$,

$$\mathrm{M}(\varepsilon) = \mathrm{M}_0 + \mathrm{M}_1 \varepsilon + \cdots + \mathrm{M}_K \varepsilon^K,$$

where $\mathrm{M}_0, \mathrm{M}_1, \ldots, \mathrm{M}_K \in \mathbb{Z}^{m \times m}$. If $K = 1$, then $\mathrm{M}(\cdot)$ is called a *linear matrix game*.

Informally, the value-positivity problems consider a polynomial matrix game $\mathrm{M}(\cdot)$ and the goal is to compare $\operatorname{val} \mathrm{M}(\varepsilon)$ with $\operatorname{val} \mathrm{M}(0)$ for small positive $\varepsilon$. Intuitively, $\varepsilon$ stands for the magnitude of small perturbations in a known direction. For simplicity and without loss of generality, we assume that $\operatorname{val} \mathrm{M}(0) = \operatorname{val} \mathrm{M}_0$ is zero. We study several variants of value positivity of polynomial matrix games and the functional form problem.

**Definition 4** (Value-Positivity Problem). Given a polynomial matrix $\mathrm{M}(\cdot)$, with $\operatorname{val} \mathrm{M}(0) = 0$, the *value-positivity problem* is to determine, for each $\varepsilon \geq 0$ small enough, the existence of a strategy that ensures a reward greater or equal to zero for the row-player when playing $\mathrm{M}(\varepsilon)$. Formally, it consists of the following decision problem:

$$\exists \varepsilon_0 > 0 \quad \forall \varepsilon \in [0, \varepsilon_0] \quad \exists \boldsymbol{p}_\varepsilon \in \Delta[m] \quad \forall j \in [m] \qquad (\boldsymbol{p}_\varepsilon^\top \mathrm{M}(\varepsilon))_j \geq 0,$$

or, equivalently, deciding if there exists $\varepsilon_0$ such that, for all $\varepsilon \in [0, \varepsilon_0]$, we have that $\operatorname{val} \mathrm{M}(\varepsilon) \geq 0$. If the answer is yes, then $\mathrm{M}(\cdot)$ is called *value positive*.

**Definition 5** (Uniform Value-Positivity Problem). Given a polynomial matrix $\mathrm{M}(\cdot)$, the *uniform value-positivity problem* is to determine the existence of a fixed strategy that ensures a reward greater or equal to zero for the row-player when playing $\mathrm{M}(\varepsilon)$ for all sufficiently small $\varepsilon \ge 0$. Formally, it consists of the following decision problem:

$$\exists \boldsymbol{p}_0 \in \Delta[m] \quad \exists \varepsilon_0 > 0 \quad \forall \varepsilon \in [0, \varepsilon_0] \quad \forall j \in [m] \qquad (\boldsymbol{p}_0^\top \mathrm{M}(\varepsilon))_j \ge 0.$$

If the answer is yes, then $\mathrm{M}(\cdot)$ is called *uniform value positive*.

**Definition 6** (Functional Form Problem). Given a polynomial matrix $\mathrm{M}(\cdot)$, the *functional form problem* is to compute the function $\varepsilon \longmapsto \operatorname{val} \mathrm{M}(\varepsilon)$ and a selection of optimal strategies for the row-player $\varepsilon \longmapsto \boldsymbol{p}^*(\varepsilon)$ in an interval of the form $[0, \varepsilon_0]$ for some $\varepsilon_0 > 0$.

Note that the value function of a polynomial matrix game is known to be a continuous piecewise rational function (see Theorem 2 below; because of Shapley and Snow [42]). Therefore, a solution to the functional form problem is encoded by rational functions $R$ and $\boldsymbol{p}^*$ such that, for all $\varepsilon \in [0, \varepsilon_0]$,

$$\operatorname{val} \mathrm{M}(\varepsilon) = R(\varepsilon),$$

and $\boldsymbol{p}^*(\varepsilon)$ is an optimal strategy of the matrix game $\mathrm{M}(\varepsilon)$.

The functional form of the value function $\varepsilon \longmapsto \operatorname{val} \mathrm{M}(\varepsilon)$ has been studied by Mills [33]. Specifically, Mills focused on computing the right derivative of the value of a perturbed matrix game, and therefore, only linear matrix games were considered.

**Definition 7** (Right Derivative of the Value Problem). Given a linear matrix $\mathrm{M}(\varepsilon) = \mathrm{M}_0 + \mathrm{M}_1 \varepsilon$, the *right derivative of the value problem* is to compute the right derivative of the function $\varepsilon \longmapsto \operatorname{val} \mathrm{M}(\varepsilon)$ at zero. Formally,

$$\mathcal{D} \operatorname{val} \mathrm{M}(0^+) := \lim_{\varepsilon \to 0^+} \frac{\operatorname{val} \mathrm{M}(\varepsilon) - \operatorname{val} \mathrm{M}(0)}{\varepsilon}.$$

### 2.2. Classical Results

Mills states the following result:

**Theorem 1** (Mills [33, Theorem 1]). *For linear matrix games $M(\varepsilon) = M_0 + M_1(\varepsilon)$ the following assertions hold:*

1. Right-derivative characterization.

$$\mathcal{D} \operatorname{val} M(0^+) = \max_{\boldsymbol{p} \in P(M_0)} \min_{\boldsymbol{q} \in Q(M_0)} \boldsymbol{p}^\top M_1 \boldsymbol{q},$$

*where $P(M_0)$ and $Q(M_0)$ are the set of optimal strategies for the row- and column-player, respectively, for the matrix game $M_0$.*

2. Right derivative computation. *We obtain $\mathcal{D} \operatorname{val} M(0^+)$ by solving an explicit LP twice the size of $M(\cdot)$.*

Recall that for a matrix $\mathrm{M} \in \mathbb{R}^{m \times m}$, its cofactor of index $(i, j) \in [m] \times [m]$ is the determinant of the matrix obtained by deleting row $i$ and column $j$ from M if $m > 1$, and one otherwise. The following result because of Shapley and Snow is fundamental in our setting.

**Theorem 2** (Value Characterization (Shapley and Snow [42])). *Consider a matrix game $M$. There exists a square submatrix $\overline{M}$ such that $\mathbf{1}^\top \operatorname{co}(\overline{M})\mathbf{1} \neq 0$ and*

$$\operatorname{val} M = \frac{\det \overline{M}}{\mathbf{1}^\top \operatorname{co}(\overline{M})\mathbf{1}}, \qquad \overline{p}^{*\top} = \frac{\mathbf{1}^\top \operatorname{co}(\overline{M})}{\mathbf{1}^\top \operatorname{co}(\overline{M})\mathbf{1}},$$

*where $\mathbf{1}$ is the vector of ones of the corresponding size, $\operatorname{co}(\overline{M})$ is the matrix of the cofactors of $\overline{M}$, and $\overline{\boldsymbol{p}}^*$ is an optimal strategy of $M$ when extended by zeros in the missing coordinates.*

### 2.3. Our Contribution

Our main contribution is the following.

**Theorem 3.** *We present polynomial-time algorithms for the value-positivity, uniform value-positivity, and functional form problems.*

**Significance.** By computing the functional form, we extend the result of Mills (which computes only the right derivative) and Shapley and Snow (which computes the value of unperturbed matrix games). Moreover, we solve the above-mentioned value-positivity problems, which have a clear game-theoretical interpretation. Section 4 is dedicated to the proof of Theorem 3.

## 3. Examples and Implications

We clarify the relationship between value positive and uniform value positive and their connection with the right derivative.

**Theorem 4** (Strict Implications of Problems). *Consider a polynomial matrix game $M(\cdot)$.*

1. *If $M(\cdot)$ is uniform value positive, then it is value positive, but the converse does not hold.*
2. *For linear matrices $M(\varepsilon) = M_0 + M_1\varepsilon$,*
   a. *if $M(\cdot)$ is value positive, then $\mathcal{D}$ val $M(0^+) \geq 0$, but the converse does not hold.*
   b. *if $M(\cdot)$ is uniform value positive, then $\mathcal{D}$ val $M(0^+) \geq 0$, but the converse does not hold.*
   c. *if $\mathcal{D}$ val $M(0^+) > 0$, then $M(\cdot)$ is value positive, but the converse does not hold.*
   d. *$\mathcal{D}$ val $M(0^+) > 0$ does not imply that $M(\cdot)$ is uniform value positive.*

**Proof.** We prove the affirmative implications and the following examples show the implications that do not hold.

1. Assume that $\mathrm{M}(\cdot)$ is uniform value positive. Then, there exists a strategy $\boldsymbol{p}_0 \in \Delta[m]$ that ensures a positive value for small $\varepsilon \geq 0$. Because $\boldsymbol{p}_0$ is one out of all possible strategies for the row-player, we have that val $\mathrm{M}(\varepsilon) \geq \min_{j\in[m]}(\boldsymbol{p}_0^\top \mathrm{M}(\varepsilon))_j \geq 0$, where the last inequality holds for $\varepsilon$ small enough. Therefore, $\mathrm{M}(\cdot)$ is value positive.
2. Consider a linear matrix $\mathrm{M}(\varepsilon) = \mathrm{M}_0 + \mathrm{M}_1\varepsilon$.
   a. Assume that $\mathrm{M}(\cdot)$ is value positive. Then, for $\varepsilon$ small enough, we have that val $\mathrm{M}(\varepsilon) \geq 0$. Therefore, taking the limit as $\varepsilon$ goes to zero, we have that

$$\mathcal{D}\ \mathrm{val}\ \mathrm{M}(0^+) = \lim_{\varepsilon\to 0^+} \frac{\mathrm{val}\ \mathrm{M}(\varepsilon)}{\varepsilon} \geq 0.$$

   This proves that value positivity implies a positive right derivative of the value function at zero.
   b. Assume that $\mathrm{M}(\cdot)$ is uniform value positive. By Theorem 4.1, we have that $\mathrm{M}(\cdot)$ is value positive. By Theorem 4.2(a), we have that $\mathcal{D}$ val $\mathrm{M}(0^+) \geq 0$.
   c. Assume that $\mathcal{D}$ val $\mathrm{M}(0^+) > 0$. Because val $\mathrm{M}(\cdot)$ is a smooth function in some right neighborhood of zero, $\mathcal{D}$ val $\mathrm{M}(0^+) > 0$ implies that val $\mathrm{M}(\varepsilon) > 0$ for $\varepsilon$ small enough. Therefore, $\mathrm{M}(\cdot)$ is value positive. □

A single example shows that the implications in items 1, 2(b), and 2(c) of Theorem 4 are tight. It consists of a linear matrix game such that (i) the right derivative of the value function is zero; that is, $\mathcal{D}$ val $\mathrm{M}(0^+) = 0$; (ii) it is value positive; and (iii) it is not uniform value positive.

**Example 1** (One-Way Implications). Consider the following linear matrix game with two actions per player.

$$\mathrm{M}(\varepsilon) = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & -3 \\ 0 & 2 \end{pmatrix}\varepsilon.$$

For every $\varepsilon > 0$, the unique optimal strategy for the row-player is given by

$$\boldsymbol{p}_\varepsilon = \left(\frac{1+\varepsilon}{2+3\varepsilon}, \frac{1+2\varepsilon}{2+3\varepsilon}\right)^\top.$$

Therefore,

$$\mathrm{val}\ \mathrm{M}(\varepsilon) = \frac{\varepsilon^2}{2+3\varepsilon}.$$

Hence, $\mathcal{D}$ val $\mathrm{M}(0^+) = 0$, and $\mathrm{M}(\cdot)$ is value positive. However, note that there is no fixed strategy $\boldsymbol{p}_0$ such that, for all small $\varepsilon > 0$, $\boldsymbol{p}_0$ guarantees a positive value for the row-player. Indeed, consider some strategy $\boldsymbol{p} \in \Delta[2]$. Then, either the probability of playing the top row under $\boldsymbol{p}$ is strictly less than $1/2$, in which case the column-player can choose the left column to ensure a negative payoff, or the probability of playing the top row is at least $1/2$, in which case the column-player can choose the right column to ensure a negative payoff for every $\varepsilon > 0$. Therefore, the polynomial matrix game $\mathrm{M}(\cdot)$ is not uniform value positive.

The following example shows that the implication in item 2-(a) of Theorem 4 is tight. It consists of a linear matrix game such that (i) the right derivative is zero, and (ii) it is not value

**Example 2** (Zero Derivative But Not Value Positive). Consider the following linear matrix game with two actions per player.

$$\mathrm{M}(\varepsilon) = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 0 & -2 \end{pmatrix}\varepsilon.$$

For every $\varepsilon > 0$, the unique optimal strategy for the row-player is given by

$$p_\varepsilon = \left(\frac{1-\varepsilon}{2-\varepsilon}, \frac{1}{2-\varepsilon}\right)^\top.$$

Therefore,

$$\text{val } \mathrm{M}(\varepsilon) = \frac{-\varepsilon^2}{2-\varepsilon}.$$

Hence, $\mathcal{D}$ val $\mathrm{M}(0^+) = 0$, and $\mathrm{M}(\cdot)$ is not value positive.

The following example proves item 2(d) of Theorem 4. It consists of a linear matrix game such that (i) the right derivative is strictly positive, and (ii) it is not uniform value positive.

**Example 3** (Tightness of Results). Consider the following linear matrix game with two actions per player.

$$\mathrm{M}(\varepsilon) = \begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix} + \begin{pmatrix} 2 & -1 \\ 0 & 0 \end{pmatrix}\varepsilon.$$

For every $\varepsilon > 0$, the unique optimal strategy for the row-player is given by

$$p_\varepsilon = \left(\frac{2}{2+3\varepsilon}, \frac{3\varepsilon}{2+3\varepsilon}\right)^\top,$$

and the value function is

$$\text{val } \mathrm{M}(\varepsilon) = \frac{\varepsilon}{2+3\varepsilon}.$$

Hence, $\mathrm{M}(\cdot)$ is value positive, and $\mathcal{D}$ val $\mathrm{M}(0^+) = 1/2 > 0$. However, $\mathrm{M}(\cdot)$ is not uniform value positive. This completes the proof of Theorem 4.

### Implications

Examples 1, 2, and 3 show the following interesting implications.

1. The value-positivity and uniform value-positivity problems are different, and they are not captured by the right derivative provided by Mills [33].
2. Even for linear matrix games, the value function can be quadratic; see Example 1. In fact, for linear matrix games of size $m \times m$, the value function can be of order $m$. Therefore, again, the right derivative solves neither the value positivity nor the uniform value-positivity problems.

Because the existing approaches of Mills [33] do not solve the value-positivity problems, we consider algorithms for them in Section 4.

## 4. Algorithms

In this section, we present polynomial-time algorithms for the value-positivity, uniform value-positivity, and functional form problems.

### 4.1. Value Positivity

The value-positivity problem consists of recognizing polynomial matrix games whose value is positive in some right neighborhood of zero.

**Main Algorithmic Idea.** The key insight is that because the polynomials have integer coefficients, the value function is a continuous, piecewise rational function; see Theorem 2. In particular, there is a root separation, so it is sufficient to know the sign of the value function at some $\varepsilon_0 > 0$ small enough. Algorithm 1 computes the small parameter $\varepsilon_0$ and the value val $\mathrm{M}(\varepsilon_0)$, and it decides whether the polynomial matrix game is value positive or not.

**Algorithm 1** (Value-Positivity Algorithm)

**Input:** $\mathrm{M}(\cdot)$ polynomial matrix game of degree $K$ and size $m \times m$
**Output:** *true* if the polynomial matrix game is value-positive; *false* otherwise

```
1: B ← max_{i,j,k}{|M_k^{i,j}|}                    ▷ Maximum absolute entry
2: ε_0 ← (2m^5 K^2 B)^{-11m^2 K}     ▷ The sign of the value function does not change in [0, ε_0]
3: if val M(ε_0) ≥ 0 then
4:     return true
5: end if
6: return false
```

**Lemma 1** (Correctness and Complexity of Algorithm 1). *Given a polynomial matrix game $M(\cdot)$, Algorithm 1 returns true if and only if $M(\cdot)$ is value positive. Moreover, it runs in polynomial time.*

Proof of Lemma 1 is given in Section 4.4.

### 4.2. Functional Form

The functional form problem consists of computing the functions $\varepsilon \longmapsto \operatorname{val} \mathrm{M}(\varepsilon)$ and $\varepsilon \longmapsto \boldsymbol{p}^*(\varepsilon)$, where $\boldsymbol{p}^*(\varepsilon)$ is an optimal strategy of the matrix game $\mathrm{M}(\varepsilon)$, in some right neighborhood of zero. Note that if there is more than one optimal strategy, then the problem imposes no restriction on which optimal strategy should be returned.

**Main Algorithmic Idea.** By Theorem 2, these two functions are piecewise rational of bounded degree. Therefore, we only need to compute their coefficients. One can recover their coefficients from evaluations by solving a system of linear equations (Jacobi [23]). Indeed, consider a rational function where the numerator and denominator have degrees $N_d$ and $D_d$, respectively,

$$f(\varepsilon) = \frac{a_0 + a_1\varepsilon + \cdots + a_{N_d}\varepsilon^{N_d}}{1 + b_1\varepsilon + \cdots + b_{D_d}\varepsilon^{D_d}}.$$

Then, the coefficients $a_0, a_1, \ldots a_{N_d}$ and $b_1, b_2, \ldots b_{D_d}$ can be computed by fitting a linear model of the form

$$y = a_0 + a_1 x + \cdots + a_{N_d} x^{N_d} - b_1 xy - \cdots - b_{D_d} x^{D_d} y.$$

This model has $(N_d + 1 + D_d)$ parameters. To fit this model and obtain the coefficients of the rational function, we only require $(N_d + 1 + D_d)$ evaluations, that is, pairs of points $(x_i, f(x_i))_{i\in[N_d+1+D_d]}$. On the one hand, for the value function, this reconstruction can be done from sufficiently many pairs of points $(\varepsilon, \operatorname{val} \mathrm{M}(\varepsilon))$. On the other hand, for optimal strategies, to compute points of a single rational function, we have to deal with one more problem, namely, the selection of the optimal strategy. We solve this problem by computing Shapley-Snow kernels, which also will provide us bounds on the degree and size of the coefficients of the model we fit.

**Definition 8** (Shapley-Snow Kernel). Consider a matrix game M of size $m \times m$. Then, every pair of sets of actions for each player $(\overline{I}, \overline{J})$, where $\overline{I} \subseteq [m]$ and $\overline{J} \subseteq [m]$, induces a smaller matrix game $\overline{\mathrm{M}}$ where players are restricted to choose an action from $\overline{I}$ and $\overline{J}$, respectively. A pair $(\overline{I}, \overline{J})$ is called a *Shapley-Snow kernel* if it induces a matrix game $\overline{\mathrm{M}}$ such that $\mathbf{1}^\top \operatorname{co}(\overline{\mathrm{M}})\mathbf{1} \neq 0$, and

$$\operatorname{val} \mathrm{M} = \frac{\det \overline{\mathrm{M}}}{1^\top \operatorname{co}(\overline{\mathrm{M}})1}, \qquad \overline{\boldsymbol{p}}^* = \frac{\operatorname{co}(\overline{\mathrm{M}})}{\mathbf{1}^\top \operatorname{co}(\overline{\mathrm{M}})\mathbf{1}}\mathbf{1},$$

where $\mathbf{1}$ is the vector of ones of the corresponding size, $\operatorname{co}(\overline{\mathrm{M}})$ is the matrix of cofactors of $\overline{\mathrm{M}}$, and $\overline{\boldsymbol{p}}^*$ is the unique optimal strategy for the matrix game $\overline{\mathrm{M}}$.

**Asymptotic Kernel.** By Theorem 2, every matrix game has at least one Shapley-Snow kernel. Moreover, there exists some pair of subsets that is a Shapley-Snow kernel for every $\varepsilon > 0$ sufficiently small. Indeed, consider the perturbed matrix game $\mathrm{M}(\cdot)$. For each $\varepsilon > 0$, the matrix game $\mathrm{M}(\varepsilon)$ has a Shapley-Snow kernel associated to a submatrix $\overline{\mathrm{M}(\varepsilon)}$, and

$$\operatorname{val} \mathrm{M}(\varepsilon) = \frac{\det \overline{\mathrm{M}(\varepsilon)}}{1^\top \operatorname{co}(\overline{\mathrm{M}(\varepsilon)})1}.$$

Note that there are at most $(2^m - 1) \cdot (2^m - 1)$ rational functions of the form $\varepsilon \longmapsto \frac{\det \overline{\mathrm{M}(\varepsilon)}}{1^\top \operatorname{co}(\overline{\mathrm{M}(\varepsilon)})1}$, one for each possible Shapley-Snow kernel. Moreover, two different such functions can coincide in only finitely many points. Therefore, because the function $\operatorname{val} \mathrm{M}(\cdot)$ is continuous, there exists a fixed pair $(\overline{I}, \overline{J})$ that is a Shapley-Snow kernel for every $\varepsilon > 0$ sufficiently small. By computing this pair, we obtain a selection of optimal strategies for every $\varepsilon ge 0$ sufficiently small.

**Computing a Kernel.** If the unperturbed matrix game has several optimal strategies, then it also necessarily has several Shapley-Snow kernels. Computing a Shapley-Snow kernel of a matrix game reduces to finding a basic solution for an LP. A polynomial-time algorithm that computes a basic solution from an arbitrary solution of an LP is presented in Karloff [26, proof of theorem 10, pp. 18–20]. Furthermore, computing an optimal basis for an LP given a pair of optimal primal and dual solutions is strongly polynomial (Megiddo [31]). Therefore, a basic solution of an LP, and so a Shapley-Snow kernel, can be found in polynomial time.

**Efficiency of Fitting.** To solve the functional form problem, we need to compute the coefficients of the corresponding rational functions. We do so by fitting a rational function to sufficiently many points. This is formalized in Algorithm 2. An alternative procedure to solve the functional form problem would be to first obtain a pair $(\overline{I},\overline{J})$ that is a Shapley-Snow kernel for all $\varepsilon$ small enough and then use Definition 8 to compute it. This alternative procedure involves computing the determinant of a matrix of polynomials, which requires exponential time in general. Therefore, fitting a rational function is more efficient.

**Algorithm 2** (Functional Form)

**Input:** $\mathrm{M}(\cdot)$ polynomial matrix game of degree $K$ and size $m \times m$
**Output:** Analytical form of the value function and an optimal strategy in some right neighborhood of zero

$B \leftarrow \max_{i,j,k}\{|\mathrm{M}_k^{i,j}|\}$
$\varepsilon_0 \leftarrow (2m^5K^2B)^{-11m^2K}$
$\boldsymbol{x} \leftarrow \left(\varepsilon_0, \frac{\varepsilon_0}{2}, \ldots, \frac{\varepsilon_0}{2mK+1}\right)$
$\boldsymbol{y} \leftarrow \left(\mathrm{val}\,\mathrm{M}(\varepsilon_0), \mathrm{val}\,\mathrm{M}\left(\frac{\varepsilon_0}{2}\right), \ldots, \mathrm{val}\,\mathrm{M}\left(\frac{\varepsilon_0}{2mK+1}\right)\right)$
val $\leftarrow$ Rational fit with numerator and denominator of degree at most $mK$ from $(\boldsymbol{x}, \boldsymbol{y})$
$(\overline{I},\overline{J}) \leftarrow$ Shapley-Snow kernel of $\mathrm{M}(\varepsilon_0)$
$\overline{\mathrm{M}}(\cdot) \leftarrow$ square submatrix corresponding to $(\overline{I},\overline{J})$
**for** $i \in [2mK+1]$ **do**
    $z_i \leftarrow$ Optimal solution of $P_{\overline{\mathrm{M}}\left(\frac{\varepsilon_0}{i}\right)}$
**end for**
$\overline{\boldsymbol{p}}^* \leftarrow$ Rational fit with numerator and denominator of degree at most $mK$ from $(x, z)$
$\boldsymbol{p}^* \leftarrow$ extension of $\overline{\boldsymbol{p}}^*$ by zero
**return** val, $p^*$

**Lemma 2** (Correctness and Complexity of Algorithm 2). *Given a polynomial matrix game* $\mathrm{M}(\cdot)$, *Algorithm 2 returns the functional form of the value and an optimal strategy of* $\mathrm{M}(\cdot)$ *in some right neighborhood of zero. Moreover, it runs in polynomial time.*

Proof of Lemma 2 is given in Section 4.4.

### 4.3. Uniform Value Positivity

The uniform value-positivity problem consists of recognizing polynomial matrix games with a fixed strategy $p_0$ for the row-player that guarantees a positive value in some right neighborhood of zero. In other words, decide if the row-player has a strategy that, for all $\varepsilon \geq 0$ small enough, guarantees at least the value of the unperturbed matrix game.

**Leading Coefficient.** An equivalent characterization of a strategy $p_0$ that guarantees uniform value positivity is as follows. First, note that a strategy $\boldsymbol{p}_0$ for the row-player and a pure action of the column-player $j \in [m]$ determine a polynomial payoff $\varepsilon \longmapsto (\boldsymbol{p}_0^\top \mathrm{M}(\varepsilon))_j$. The sign of this polynomial in some right neighborhood of zero is determined by its *leading coefficient*, that is, the first nonzero coefficient ordered by the degree of the corresponding monomial it multiplies. Therefore, a witness of uniform value positivity is a strategy $p_0$ for which all the corresponding polynomial payoffs (one for each action of the column-player) either have a strictly positive leading coefficient or are constant to zero.

**Frontier.** Consider a polynomial matrix game $\mathrm{M}(\cdot)$, and assume it is uniform value positive. Note that, if for some action $j$ of the column-player, the corresponding polynomial payoff $\varepsilon \longmapsto (\boldsymbol{p}_0^\top \mathrm{M}(\varepsilon))_j$ has a leading coefficient of order, for example, 3, then changing the perturbation coefficients at column $j$ of order 4 or more does not change the fact that the polynomial matrix game is uniform value positive. Therefore, for each witness strategy $\boldsymbol{p}_0$ for $\mathrm{M}(\cdot)$, there is a sequence of indices (one per column) containing the corresponding leading coefficients' order of perturbation. We represent these indices by a vector $\mathbf{k} \in [K+1]^m$, where the index $(K+1)$ corresponds to the polynomial payoff being constant to zero. We call such a sequence a *frontier*.

A useful visualization of the frontier of a strategy $p$ of the row-player is the following. Consider the matrix given by $((\boldsymbol{p}^\top \mathrm{M}_k)_j)_{k,j\in[m]}$. In other words, each column corresponds to an action of the column-player, and each row corresponds to the coefficient of the respective polynomial when the row-player plays according to $\boldsymbol{p}$. The leading coefficient of the strategy $\boldsymbol{p}$ on a column corresponds to the first (as $k$ increases) nonzero entry, if there is some. The frontier of $\boldsymbol{p}$ represents the indices where the leading coefficients are in this matrix. Note that, if all

**Figure 1.** Illustration of polynomials given a strategy. Illustration of $((\boldsymbol{p}^\top \mathrm{M}_k)_j)_{k,j\in[m]}$ for a hypothetical example with $K=2$ and $m=3$. Highlighted entries represent the frontier of this strategy, $\mathbf{k}=(3,1,2)$. The first column has no leading coefficient, as it is full of zeros. The second column has a leading coefficient of two. The third column has a leading coefficient of $-1$.

$$
\begin{array}{c}
\begin{array}{cccc} & 1 & 2 & 3 \end{array}\\
\begin{array}{c} M_0 \\ M_1 \\ M_2 \\ M_3 \end{array}
\begin{pmatrix}
0 & 0 & 0\\
0 & \boxed{2} & 0\\
0 & -1 & \boxed{-1}\\
\boxed{0} & 0 & 0
\end{pmatrix}.
\end{array}
$$

columns of this matrix have a strictly positive leading coefficient or are full of zeros, then $p$ guarantees the uniform value positivity of $\mathrm{M}(\cdot)$. Figure 1 visualizes the matrix given by $((\boldsymbol{p}^\top \mathrm{M}_k)_j)_{k,j\in[m]}$ in a hypothetical example.

**Verification of a Frontier.** Given a frontier, deciding if there is a strategy $p_0$ with this frontier and positive leading coefficients is polynomial time: it consists of solving at most $m$ LPs. Indeed, for a frontier $\mathbf{k}\in[K+1]^m$ and an action of the column-player $j_0\in[m]$, which we call *focus action*, define the following LP.

$$
(P_{\mathbf{k},j_0})\begin{cases}
\max\limits_{p} & (\boldsymbol{p}^\top \mathrm{M}_{\mathbf{k}_{j_0}})_{j_0}\\
s.t. & (\boldsymbol{p}^\top \mathrm{M}_k)_j \ge 0, \qquad \forall j\in[m], k\le \mathbf{k}_j\,\cdot\\
 & \quad \boldsymbol{p}\in\Delta([m])
\end{cases}
$$

Note that if $(P_{\mathbf{k},j_0})$ is feasible, then its value is at least zero. Figure 2 visualizes the constraints and objective function in $(P_{\mathbf{k},j_0})$. The LP $(P_{\mathbf{k},j_0})$ computes a strategy whose frontier is at least $\mathbf{k}$ and maximizes the candidate leading coefficient of the polynomial corresponding to the focus action $j_0$. To verify if a given frontier has a witness strategy, we do as follows.

1. Check if there is a strategy making all the coefficients up to the frontier at least zero, discarding the frontier if there is none.
2. Iterate over all focus actions, maximizing the corresponding coefficient in the frontier. For each focus action, the maximum value of the corresponding coefficient is either zero or strictly positive.
3. If all focus actions returned a strictly positive value, then there is a witness for the frontier. If some focus action returned zero (and $\mathbf{k}_{j_0}<K+1$), then there is no witness with this frontier.

**Lexicographic Search.** The problem with frontiers is that there are $(K+1)^m$ of them, that is, exponentially many. Therefore, we must search efficiently among all possible frontiers. Algorithm 3 implements a search pattern that considers at most $(K+1)m$ frontiers. During this search pattern, the coordinates of the candidate frontier are always increasing, exploiting properties of the verification of a frontier. Indeed, while verifying a frontier, if a focus action returns zero, then the search can safely continue to the next frontier in our search pattern. Altogether, the algorithm decides whether the polynomial matrix game is uniform value positive or not by solving at most $(K+1)m^2$ LPs. This is formally presented in Algorithm 3.

**Figure 2.** Illustration of $(P_{\mathbf{k},j_0})$ for a hypothetical frontier $\mathbf{k}=(3,1,2)$ and focus column $j_0=2$. The coordinates with entries $\ge 0$ correspond to the constraints of the program, that is, coordinates $(k,j)\in\{1,2,3\}\times\{0,1,2,3\}$ such that $k\le \mathbf{k}_j$. The highlighted entry at the coordinate $(\mathbf{k}_{j_0}, j_0)$ corresponds to the objective function of the program.

$$
\begin{array}{c}
\begin{array}{cccc} & 1 & 2 & 3 \end{array}\\
\begin{array}{c} M_0 \\ M_1 \\ M_2 \\ M_3 \end{array}
\begin{pmatrix}
\ge 0 & \ge 0 & \ge 0\\
\ge 0 & \boxed{\ge 0} & \ge 0\\
\ge 0 & & \ge 0\\
\ge 0 & & 
\end{pmatrix}
\end{array}
$$

**Algorithm 3** (Uniform Value-Positivity Algorithm)
**Input:** M$(\cdot)$ polynomial matrix game of degree $K$ and size $m \times m$
**Output:** *true* if the polynomial matrix game is uniform value-positive; *false* otherwise.

1: $\mathbf{k} \leftarrow (0, 0, \ldots, 0) \in [K+1]^m$ ▷ Initialize the frontier
2: **loop**
3: **if** $(P_{\mathbf{k},1})$ is infeasible **then** ▷ Test feasibility of the frontier
4: **return** false ▷ OUTPUT: There is no witness
5: **end if**
6: **for** $j_0 \in [m]$ **do** ▷ Iterate over column-player's actions
7: **if** $\mathbf{k}_{j_0} = K+1$ **then** ▷ This action has been fully investigated
8: **continue** the for loop ▷ Consider the next focus action
9: **end if**
10: **if** val $(P_{\mathbf{k},j_0}) == 0$ **then** ▷ Higher order terms need to be investigated
11: $\mathbf{k} \leftarrow \mathbf{k} + e_{j_0}$ ▷ Advance the frontier
12: **goto** to Line 3 ▷ Investigate the new frontier
13: **end if**
14: **if** val $(P_{\mathbf{k},j_0}) > 0$ **then** ▷ The focus action is covered
15: **continue** the for loop ▷ Consider the next focus action
16: **end if**
17: **end for**
18: **return** true ▷ OUPUT: The frontier has a witness strategy
19: **end loop**

**Lemma 3** (Correctness and Complexity of Algorithm 3). *Given a polynomial matrix game $M(\cdot)$, Algorithm 3 returns true if and only if $M(\cdot)$ is uniform value positive. Moreover, it runs in polynomial time.*

Proof of Lemma 3 is given in Section 4.4.

**Example 4** (Running the Uniform Value-Positivity Algorithm). Consider the following polynomial matrix game:

$$\mathrm{M}(\varepsilon) = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & -3 \\ 0 & 2 \end{pmatrix} \varepsilon.$$

This polynomial matrix game is not uniform value positive. Indeed, $M_0$ has a unique optimal strategy $\boldsymbol{p}^* = (1/2, 1/2)^\top$, which, for all $\varepsilon > 0$, does not guarantee zero in $M(\varepsilon)$. Let us show how Algorithm 3 arrives at this conclusion. Recall that Algorithm 3 iterates over frontiers to make a decision. For each frontier, it makes a feasibility check and then checks each focus action. For each focus action, if the value of the related LP is zero, then it advances the frontier. If the value is strictly positive, it continues to the next focus action. During this procedure, each coordinate of the frontier increases. We now go over this procedure explicitly.

**Frontier (0, 0), Action 1.** The initial frontier is $\mathbf{k} = (0, 0)$. Write a strategy $p$ as $(x, 1-x)^\top$. The first LP to check feasibility is the following:

$$(P_{(0,0),1}) \begin{cases} \max\limits_{x} & x \cdot 1 + (1-x) \cdot (-1) \\ s.t. & x \cdot 1 + (1-x) \cdot (-1) \geq 0 \\ & x \cdot (-1) + (1-x) \cdot 1 \geq 0 \\ & x \in [0, 1]. \end{cases}$$

This LP is feasible, so we proceed to check each focus action. The first column-player's action we investigate is $j_0 = 1$. We compute the value of $(P_{(0,0),1})$. Its only solution is $\boldsymbol{p}^* = (1/2, 1/2)^\top$, and its value is zero. Therefore, we increase the frontier by one in its $j_0$-th coordinate. The new frontier is $\mathbf{k} = (1, 0)$.

**Frontier (1, 0), Action 1.** The second LP to consider is the following:

$$(P_{(1,0),1}) \begin{cases} \max\limits_{x} & x \cdot 1 \\ s.t. & x \cdot 1 + (1-x) \cdot (-1) \geq 0 \\ & x \cdot 1 \geq 0 \\ & x \cdot (-1) + (1-x) \cdot 1 \geq 0 \\ & x \in [0, 1]. \end{cases}$$

Again, it is feasible, so we check each focus action. Considering $j_0 = 1$ and the corresponding LP, the only solution is $\boldsymbol{p}^* = (1/2, 1/2)^\top$, and its value is strictly positive. Therefore, we continue with the next action of the column-player $j_0 = 2$.

**Frontier (1, 0), Action 2.** The third LP to solve is the following:

$$(P_{(1,0),2})\begin{cases} \max\limits_x & x\cdot(-1) + (1-x)\cdot 1 \\ s.t. & x\cdot 1 + (1-x)\cdot(-1) \geq 0 \\ & x \cdot 1 \geq 0 \\ & x\cdot(-1) + (1-x)\cdot 1 \geq 0 \\ & x \in [0,1]. \end{cases}$$

Note that only the objective function changed with respect to $(P_{(1,0),1})$. Therefore, its only solution is still $\boldsymbol{p}^* = (1/2, 1/2)^\top$, and its value is zero, so we increase the frontier by one in its $j_0$-th coordinate. The new frontier is $\mathbf{k} = (1,1)$.

**Frontier (1, 1), Action 1.** The fourth LP to solve is the following:

$$(P_{(1,1),1})\begin{cases} \max\limits_x & x\cdot 1 \\ s.t. & x\cdot 1 + (1-x)\cdot(-1) \geq 0 \\ & x \cdot 1 \geq 0 \\ & x\cdot(-1) + (1-x)\cdot 1 \geq 0 \\ & x\cdot(-3) + (1-x)\cdot 2 \geq 0 \\ & x \in [0,1]. \end{cases}$$

This last LP is infeasible. Therefore, the algorithm outputs *false*, correctly indicating that this polynomial matrix game is not uniform value positive.

### 4.4. Detailed Proofs

In this section, we prove Lemma 1, Lemma 2, and Lemma 3, which jointly prove our main result, Theorem 3. We start by recalling classical results on polynomials and follow with the technical proofs.

**4.4.1. Classical Results.** We recall a classical result on the minimal separation of roots of polynomials.

**Lemma 4** (Bounding Polynomial Roots (Cauchy [13], Mahler [29])). *Consider a nonzero polynomial with integer coefficients $r(\varepsilon) = a_1\varepsilon + a_2\varepsilon^2 + \cdots + a_L\varepsilon^L$. Then, the first strictly positive root is bounded away from zero. Formally, if $r(\varepsilon_0) = 0$ and $\varepsilon_0 > 0$, then*

$$\varepsilon_0 \geq L^{-(L+2)/2} \|r\|_2^{1-L},$$

*where* $\|r\|_2^2 := \sum_{i\in[L]} a_i^2$.

We deal with polynomials that correspond to the determinant of a matrix with polynomial entries, and to apply Lemma 4, we need to bound their degree and the size of their coefficients. Denote the binary size of $x$ by bit $(x)$; that is, if $x$ is a number, it is $\lceil \log_2(|x|+1)\rceil$, and if $x$ is a collection of numbers, then it is the sum of their binary size. We recall the following classical result, which is a consequence of Basu et al. [8, proposition 8.12].

**Lemma 5** (Order of Matrix Determinant (Basu et al. [8])). *Consider a polynomial matrix game $M(\cdot)$ of size $m \times m$, order $K$, and integer entries of size bounded by $B$. The function det $M(\cdot)$ is a polynomial of degree at most $mK$ and coefficients of binary size at most $m$* bit $(M) + m$ bit $(m) +$ bit $(mK+1)$.

**4.4.2. Technical Proofs.** We show a characterization of the value function of polynomial matrix games in an explicit, at most exponentially small, right neighborhood of zero. Although similar results can be found in the literature (see Oliu-Barton [36] and Szczechla et al. [46, lemma 5.1, p. 874]), we remark on the importance of the numerical bounds for our algorithms.

**Lemma 6** (Value Characterization Close to Zero). *Consider a polynomial matrix game $M(\cdot)$ of size $m \times m$, order $K$ and integer entries of size bounded by $B$. Then, there exists a rational function $R_1$, a rational vector function $R_2$, and a threshold*

$\varepsilon_0 \geq (2m^5K^2B)^{-11m^2K}$ *such that, for all* $\varepsilon \in [0, \varepsilon_0]$,

$$\operatorname{val} M(\varepsilon) = R_1(\varepsilon) \qquad p^*(\varepsilon) = R_2(\varepsilon).$$

*Moreover, the functions* $R_1$ *and each coordinate of* $R_2$ *can be expressed as the quotient of two polynomials* $r/s$*, where* $r$ *and* $s$ *are at most of degree* $mK$*, the numerator* $r$ *has no roots in* $(0, \varepsilon_0]$ *unless it is constant to zero,* $s$ *is strictly positive in* $[0, \varepsilon_0]$*, and* $s(0) = 1$.

**Proof.** Let $\mathrm{M}(\cdot)$ be a polynomial matrix game of size $m \times m$, order $K$, and integer entries of size bounded by $B$. By Theorem 2, for every $\varepsilon \geq 0$, there exists a square submatrix $\overline{\mathrm{M}(\varepsilon)}$ such that $\mathbf{1}^\top \operatorname{co}(\overline{\mathrm{M}(\varepsilon)})\mathbf{1} \neq 0$ and

$$\operatorname{val} \mathrm{M}(\varepsilon) = \frac{\det \overline{\mathrm{M}(\varepsilon)}}{\mathbf{1}^\top \operatorname{co}(\overline{\mathrm{M}(\varepsilon)})\mathbf{1}}. \tag{1}$$

Because there are finitely many square submatrices and $\operatorname{val} \mathrm{M}(\cdot)$ is continuous, $\operatorname{val} \mathrm{M}(\cdot)$ is a piecewise rational function. Define the threshold $\varepsilon_0 := (2m^5K^2B)^{-11m^2K}$. We prove that $\operatorname{val} \mathrm{M}(\cdot)$ is a rational function in $[0, \varepsilon_0]$.

Denote $\mathcal{R}$ the set of all rational functions obtained as one varies the submatrix $\overline{\mathrm{M}}$ in Equation (1). Note that two different such rational functions intersect only finitely many times. We claim that $\varepsilon_0$ is a lower bound on the closest parameter to zero where an intersection happens. Indeed, by Lemma 5, $\det \overline{\mathrm{M}(\varepsilon)}$ and the coordinates of $\operatorname{co}(\overline{\mathrm{M}(\varepsilon)})$ are polynomials of degree at most $mK$ and coefficients of binary size at most $m \operatorname{bit}(\mathrm{M}) + m\operatorname{bit}(m) + \operatorname{bit}(mK+1)$. Therefore, for all $R \in \mathcal{R}$, we have that $R = r/s$, where $r$ and $s$ are polynomials of degree at most $mK$ and coefficients of binary size at most $m \operatorname{bit}(\mathrm{M}) + (m+2)\operatorname{bit}(m) + \operatorname{bit}(mK+1)$. Then, we have that if $R \neq R' \in \mathcal{R}$ and $\varepsilon_0 > 0$ are such that $R(\varepsilon_0) = R'(\varepsilon_0)$, then $\varepsilon_0$ is the root of a polynomial of degree at most $2mK$ whose coefficients have binary size at most $2m \operatorname{bit}(\mathrm{M}) + 2(m+2)\operatorname{bit}(m) + 4 \operatorname{bit}(mK+1)$. Then, applying Lemma 4 to $L = 2mK$ and $\log(\|r\|_2) \leq \log(2mK) + 2m \operatorname{bit}(\mathrm{M}) + 2(m+2)\operatorname{bit}(m) + 4 \operatorname{bit}(mK+1)$, we get that

$$\begin{aligned}
\log(\varepsilon_0) &> -(mK+1)\log(2mK) \\
&\qquad -4mK(m+2)(\log(2mK) + \operatorname{bit}(\mathrm{M}) + \operatorname{bit}(m) + \operatorname{bit}(mK+1)), \\
&\geq -(mK+1)\log(2mK) - 9m^2K\log(2m^5K^2B), \\
&\geq -11m^2K\log(2m^5K^2B);
\end{aligned}$$

that is, $\varepsilon_0 \geq (2m^5K^2B)^{-11m^2K}$. In conclusion, there exists $R_1 \in \mathcal{R}$ such that, for all $\varepsilon \in [0, \varepsilon_0]$,

$$\operatorname{val} \mathrm{M}(\varepsilon) = R_1(\varepsilon).$$

Moreover, either $R_1$ is constant to zero, or it has no roots in $(0, \varepsilon_0]$. The rest of the properties of $R_1$ are achieved by renormalization.

For the optimal strategy $\boldsymbol{p}^*$, recall that the submatrix $\overline{\mathrm{M}}$ is given by a pair of indices $(\bar{I}, \bar{J})$. By Theorem 2, an optimal strategy $\boldsymbol{p}^*$ is recovered as the extension by zeros outside the rows of $\bar{I}$ of the rational vector function

$$\varepsilon \longmapsto \frac{\operatorname{co}(\overline{\mathrm{M}}(\varepsilon))}{\mathbf{1}^\top \operatorname{co}(\overline{\mathrm{M}}(\varepsilon))\mathbf{1}}\mathbf{1}.$$

Therefore, $\boldsymbol{p}^*$ can be given by another rational vector function $R_2 = r/s$. The properties of $r$ and $s$ are proven the same way as for the value function. □

Lemma 6 allows us to prove Lemma 1 as follows.

**Proof of Lemma 1.** We show the correctness and complexity of Algorithm 1. By Lemma 6, the value function does not change its sign between zero and $\varepsilon_0 = (2m^5K^2B)^{-11m^2K}$. Therefore, the sign of $\operatorname{val} \mathrm{M}(\varepsilon_0)$ is the sign of the value function in some right neighborhood of zero. This proves that Algorithm 1 is correct.

Computing the value of a matrix game takes polynomial time in terms of the binary size of the matrix game. In this case, the binary size of $\mathrm{M}(\varepsilon_0)$ is at most the binary size of the polynomial matrix game plus the binary size of $\varepsilon_0^K$, which is at most $11m^2K^2\lceil\log_2(2m^5K^2B)\rceil$. Therefore, computing the value of the matrix game $\mathrm{M}(\varepsilon_0)$ takes only polynomial time, and so, Algorithm 1 runs in polynomial time. □

Before we prove Lemma 2, we clarify how Algorithm 2 performs line 6, namely, computing a Shapley-Snow kernel of a matrix game in polynomial time.

**Lemma 7.** *Consider a matrix game M of size $m \times m$ and integer entries of size bounded by B. A Shapley-Snow kernel can be computed in polynomial time in terms of m and B.*

Recall that Shapley-Snow kernels are supports of optimal strategies of minimal support size and are closely related to basic solutions of LPs (Shapley and Snow [42]), that is, solutions uniquely defined by a subset of linearly independent constraints. We compute a Shapley-Snow kernel by computing basic solutions of LPs.

**Proof of Lemma 7.** Consider a matrix game M of size $m \times m$ and integer entries of size bounded by $B$. First, we relate a basic solution of an LP to the support of an optimal strategy in the matrix game. Then, we give an elimination procedure to reduce the size of the support until we find a Shapley-Snow kernel.

Basic solutions as the support of optimal strategies. For the LP $(P_{\mathrm{M}})$, constraints correspond to either (i) restrictions given by actions of the column-player, or (ii) restrictions forcing $\boldsymbol{p}$ to be a distribution over rows. Therefore, a basic solution relates to a subset of actions of the column-player $\bar{J}$, which contains the support of an optimal strategy for the column-player, as these actions can force the value. Because $\bar{J}$ contains the support of an optimal strategy, there is a Shapley-Snow kernel $(\bar{I}', \bar{J}')$ such that $\bar{J}' \subseteq \bar{J}$. Similarly, from a basic solution of the LP $(P_{-\mathrm{M}^\top})$, we obtain a candidate set $\bar{I}$ for the row-player. Although $(\bar{I}, \bar{J})$ might not be a Shapley-Snow kernel, it always contains one. Therefore, through a suitable elimination procedure, we extract a Shapley-Snow kernel.

Elimination procedure. Consider a pair $(\bar{I}, \bar{J})$ candidate for a Shapley-Snow kernel. While eliminating actions from $\bar{I}$ or $\bar{J}$, we distinguish two cases: (i) $|\bar{I}| \neq |\bar{J}|$, and (ii) $|\bar{I}| = |\bar{J}|$, but $1^\top \mathrm{co}\,(\overline{\mathrm{M}})1 = 0$. If none of these cases hold, then $(\bar{I}, \bar{J})$ is a Shapley-Snow kernel. Therefore, we show how to eliminate actions in each case until we find a Shapley-Snow kernel.

For case (i), without loss of generality, consider $|\bar{I}| > |\bar{J}|$. Recall that $\bar{I}$ corresponds to the support of an optimal strategy. Therefore, we correctly eliminate an action from $\bar{I}$ to find a smaller support also containing an optimal strategy as follows. We solve $|\bar{I}|$ many LPs, one for each action, where the corresponding action is forced to have probability zero (and therefore eliminated from the support). Because $\bar{I}$ is not of minimal size, at least one of these LPs has the same value as the original LP. Thus, we obtain a new candidate set $\bar{I}' \subsetneq \bar{I}$. Repeating this procedure, we obtain a candidate pair $(\bar{I}, \bar{J})$ such that $|\bar{I}| = \bar{J}|$.

For case (ii), recall that $(\bar{I}, \bar{J})$ contains a Shapley-Snow kernel. Therefore, we eliminate one of the actions from $\bar{I}$ as done in case (i) to obtain a new candidate set $\bar{I}' \subsetneq \bar{I}$ that contains an optimal strategy, and we go back to case (i).

Iterating this procedure, we obtain a Shapley-Snow kernel $(\bar{I}, \bar{J})$. Following the approach in the proof of Karloff [26, theorem 10, pp. 18–20], a basic solution of $(P_{\mathrm{M}})$ can be computed in polynomial time. Therefore, computing a Shapley-Snow kernel also takes only polynomial time. □

We now proceed to Proof of Lemma 2, that is, the correctness and complexity of Algorithm 2. Recall that Algorithm 2 computes the value function by fitting $(2mK+1)$ points and an optimal strategy function by (i) computing a Shapley-Snow kernel of the matrix game $\mathrm{M}(\varepsilon_0)$, (ii) computing optimal strategies for the corresponding subgame at various values of $\varepsilon$, (iii) fitting a rational function to these optimal strategies, and (iv) extending by zeros to a strategy of the original game.

**Proof of Lemma 2.** We consider first the value function and then the optimal strategy function.

Value function. By Lemma 6, the functional form of the value function is given by a rational function with a numerator and denominator of degree at most $mK$. Therefore, its coefficients are uniquely determined by $(2mK+1)$ values, for example, its value at the points $\varepsilon_0, \varepsilon_0/2, \dots, \varepsilon_0/(2mk+1)$. The computation of this fitting is polynomial time, as it corresponds to solving a system of linear equations, which can be done in polynomial time.

Optimal strategy. By Lemma 7, $(\bar{I}, \bar{J})$ given in line 6 is a Shapley-Snow kernel of $\mathrm{M}(\varepsilon_0)$ obtained in polynomial time. By Lemma 6, for all $\varepsilon \in [0, \varepsilon_0]$, the pair $(\bar{I}, \bar{J})$ is a Shapley-Snow kernel, and an optimal strategy of the reduced game $\overline{\mathrm{M}(\varepsilon)}$ is optimal in $\mathrm{M}(\varepsilon)$ when extended by zeros, and each of its coordinates is a rational function with a numerator and denominator of degree at most mK. Therefore, their coefficients are uniquely determined by $(2mK+1)$ values, for example, their values at the points $\varepsilon_0, \varepsilon_0/2, \dots, \varepsilon_0/(2mk+1)$. Algorithm 2 computes these values, as optimal strategies of $\overline{\mathrm{M}(\varepsilon)}$ are unique by definition of the Shapley-Snow kernel. Therefore, Algorithm 2 recovers an optimal strategy of the subgame by fitting a rational function to each coordinate. Lastly, as already mentioned, extending it by zeros in the coordinates outside of $\bar{I}$ leads to an optimal strategy $\boldsymbol{p}^*(\varepsilon)$. □

Proof of Lemma 3 is more involved and requires a few more technical lemmas. We start by recalling the LP used in Algorithm 3. For a vector $\mathbf{k} \in [K]^m$ and an index $j_0 \in [m]$, we consider the following LP:

$$(P_{\mathbf{k},j_0}) \begin{cases} \max\limits_{p} & (\boldsymbol{p}^\top \mathrm{M}_{\mathbf{k}_{j_0}})_{j_0} \\ s.t. & (\boldsymbol{p}^\top \mathrm{M}_k)_j \geq 0, \qquad \forall j \in [m], k \leq \mathbf{k}_j \\ & \qquad \boldsymbol{p} \in \Delta([m]). \end{cases}$$

Algorithm 3 implements a search over frontiers and determines whether the polynomial matrix game $\mathrm{M}(\cdot)$ is uniformly value positive or not. Lemma 8 proves sufficient conditions for uniform value positivity (the same conditions that are verified by Algorithm 3). Lemma 9 relates the search over frontiers with all possible witnesses of uniform value positivity. Finally, Lemma 10 proves sufficient conditions to determine that $\mathrm{M}(\cdot)$ is not uniformly value positive. Together, these lemmas prove the correctness of Algorithm 3.

**Lemma 8.** *Consider a polynomial matrix game $M(\cdot)$ of size $m \times m$ and order $K$. Fix $\mathbf{k} \in [(K+1)]^m$. Assume that* (i) *the LP $(P_{\mathbf{k},1})$ is feasible, and* (ii) *for all $j_0 \in [m]$, if $\mathbf{k}_{j_0} < K+1$, then the value of $(P_{\mathbf{k},j_0})$ is strictly positive. Then, $M(\cdot)$ is uniform value positive.*

**Proof.** We construct the witness of uniform value positivity as a convex combination of the solutions of the LPs $(P_{\mathbf{k},j_0})$. For all $j_0 \in [m]$, because $(P_{\mathbf{k},1})$ is feasible, $(P_{\mathbf{k},j_0})$ is feasible, as only the objective function changes. Consider $\boldsymbol{p}^*(j_0)$ a solution of $(P_{\mathbf{k},j_0})$, and define

$$\boldsymbol{p} := \frac{1}{m}\sum_{j_0=1}^{m} \boldsymbol{p}^*(j_0).$$

We claim that $p$ is a witness of uniform value positivity. Indeed, on the one hand, if $\mathbf{k}_{j_0} < K+1$, then the leading coefficient of the polynomial $(\boldsymbol{p}^\top M(\varepsilon))_{j_0}$ is strictly positive because it is lower bounded by the corresponding coefficient of $\boldsymbol{p}^*(j_0)$ divided by $m$, which is strictly positive by assumption. On the other hand, if $\mathbf{k}_{j_0} = K+1$, then all coefficients of $(\boldsymbol{p}^\top M(\varepsilon))_{j_0}$ are zero. Overall, for all $j_0 \in [m]$ and $\varepsilon \geq 0$ sufficiently small, $(p^\top M(\varepsilon))_{j_0}$ is positive, so $\mathrm{M}(\cdot)$ is uniform value positive. □

Algorithm 3 maintains an invariant while increasing the coordinates of $\mathbf{k}$, formalized as follows:

**Lemma 9.** *Consider a polynomial matrix game $M(\cdot)$ of size $m \times m$ and order $K$ that is uniform value positive with a witness strategy $\boldsymbol{p}$. Then, for all frontiers $\mathbf{k} \in [(K+1)]^m$ obtained during the execution of Algorithm 3, we have that, for all $j \in [m]$ and $0 \leq k < \mathbf{k}_j$, it holds that $(\boldsymbol{p}^\top M_k)_j = 0$.*

**Proof.** The proof is by induction on the frontier $\mathbf{k} \in [K+1]^m$. For the initial case, we have $\mathbf{k} = (0, 0, \ldots, 0)$, and the statement is trivially true because the condition is vacuous: for all $j \in [m]$, there is no $k$ such that $0 \leq k$ and $k < \mathbf{k}_j = 0$. We continue to the inductive step.

Assume the statement is true for $\mathbf{k}$. We will prove that the statement holds for the next frontier $\mathbf{k}'$ obtained during the execution of Algorithm 3. The only way to obtain $\mathbf{k}'$ is that there exists $j_0 \in [m] \setminus \{j : \mathbf{k}_j = K+1\}$ such that the value of $(P_{\mathbf{k},j_0})$ is zero. In this case, the next frontier is $\mathbf{k}' := \mathbf{k} + \boldsymbol{e}_{j_0}$, the indicator vector of the coordinate $j_0$. Consider $\boldsymbol{p}$ a witness of the uniform value positivity of $\mathrm{M}(\cdot)$; that is, for all $j \in [m]$ and all $\varepsilon$ small enough, we have that $(\boldsymbol{p}^\top M(\varepsilon))_j \geq 0$. Equivalently, for all $j \in [m]$, the leading coefficient of $(\boldsymbol{p}^\top M(\varepsilon))_j$ is strictly greater than zero, or the polynomial is constantly zero. By the inductive hypothesis, we know that for all $j \in [m]$ and all $0 \leq k < \mathbf{k}_j$ we have that $(\boldsymbol{p}^\top M_k)_j = 0$. Therefore, we have that $(\boldsymbol{p}^\top M_{\mathbf{k}_{j_0}})_{j_0} \geq 0$. We only need to prove that $(\boldsymbol{p}^\top M_{\mathbf{k}_{j_0}})_{j_0} = 0$.

Note that $p$ is a feasible vector of $(P_{\mathbf{k},j_0})$, whose value is zero. Therefore, $(\boldsymbol{p}^\top M_{\mathbf{k}_{j_0}})_{j_0} = 0$ must hold. This concludes the inductive step. □

Lastly, we study what happens when $(P_{\mathbf{k},1})$ is infeasible.

**Lemma 10.** *Consider a polynomial matrix game $M(\cdot)$ of size $m \times m$ and order $K$, and $\mathbf{k} \in [(K+1)]^m$, obtained during the execution of Algorithm 3. If $(P_{\mathbf{k},1})$ is infeasible, then $M(\cdot)$ is not uniform value positive.*

**Proof.** Assume that $(P_{\mathbf{k},1})$ is infeasible. By contradiction, assume that $\mathrm{M}(\cdot)$ is uniform value positive, and consider a witness strategy $\boldsymbol{p}$. By Lemma 9, we have that, for all $j \in [m]$ and all $0 \leq k < \mathbf{k}_j$, it holds that $(\boldsymbol{p}^\top M_k)_j = 0$. Because the strategy $p$ is a witness of uniform value positivity, for all $j \in [m]$, the polynomial $(\boldsymbol{p}^\top M(\varepsilon))_j$ is positive in some right neighborhood of zero. Therefore, for all $j \in [m]$, it holds that $(\boldsymbol{p}^\top M_{\mathbf{k}_j})_j \geq 0$. In conclusion, $p$ is a feasible vector of $(P_{\mathbf{k},1})$, which is a contradiction. □

We now prove Lemma 3.

**Proof of Lemma 3.** Note that Algorithm 3 solves at most $(K+1)m^2$ LPs, terminating in polynomial time. Moreover, by Lemma 10, we have that whenever Algorithm 3 returns *false*, the polynomial matrix game $\mathrm{M}(\cdot)$ is not uniform value positive.

Lastly, note that if, during the execution of Algorithm 3, there exists $j_0 \in [m]$ such that $\mathbf{k}_{j_0} = K+1$, then $(P_{\mathbf{k},j_0})$ is feasible because $M_{K+1}$ is the matrix of only zeros. Therefore, whenever Algorithm 3 returns *true*, all conditions of Lemma 8 are satisfied, so $\mathrm{M}(\cdot)$ is uniform value positive. □

## 5. Value Positivity and Perturbed LPs

In this section, we define *robustness problems* for perturbed LPs and connect them to value positivity for polynomial matrix games.

### 5.1. Perturbed LPs and Robustness Problems

The notion of robust LPs has been considered in many different contexts, and some examples are the following. First, LPs are closely related to Markov decision processes (MDPs), and robust MDPs consider that the transition function is not known but comes from an uncertain set, which leads to a notion of robust LPs (Nilim and El Ghaoui [34]). Second, analytical perturbation of LPs has been considered in Avrachenkov et al. [7, chapter 5]. Finally, considering perturbations of inputs in a neighborhood and efficient algorithmic solutions of LPs via simplex leads to the notion of smoothed complexity analysis (Spielman and Teng [45]). There is a vast literature on robust LPs, and in this work, we focus on LPs in connection to matrix games. Whereas Mills [33] considered only linear perturbations, we consider general polynomial perturbations.

**Definition 9** (Perturbed LPs). A perturbed LP of degree $K$ and size $m \times m$ has the following form:

$$(P(\varepsilon))\begin{cases} \max\limits_{\boldsymbol{x}} & (\boldsymbol{c}_0 + \boldsymbol{c}_1\varepsilon + \cdots + \boldsymbol{c}_K\varepsilon^K)^\top \boldsymbol{x} \\ s.t. & (\mathrm{A}_0 + \mathrm{A}_1\varepsilon + \cdots + \mathrm{A}_K\varepsilon^K)\boldsymbol{x} \le \boldsymbol{b}_0 + \boldsymbol{b}_1\varepsilon + \cdots + \boldsymbol{b}_K\varepsilon^K \\ & \boldsymbol{x} \ge 0, \end{cases}$$

where $\mathrm{A}_i$, $\boldsymbol{b}_i$ and $\boldsymbol{c}_i$ have integer entries and $\mathrm{A}_i$ is of size $m \times m$, for all $i \in [K]$.

**Robustness Problems.** The stability of LPs concerns the robustness of the solutions and the value upon perturbations. Assuming that $(P_0)$ is a feasible and bounded LP, *robustness problems* consist of determining how its feasibility, optimal solutions, and value vary for small positive $\varepsilon$.

Perturbed LPs can lead to irregularities; namely, discontinuities may arise at $\varepsilon = 0$ even when perturbations are regular and continuous; see Avrachenkov et al. [7, chapter 5, p. 111] for examples. Therefore, studying robustness on LPs usually requires structural assumptions on the perturbations, such as the rank of $\mathrm{A}(\varepsilon)$ being constant near zero. In our case, we focus on perturbations where both primal and dual admit uniformly bounded feasible regions. We formalize this assumption in Section 5.2. In the sequel, $(Q)$ denotes the dual LP of $(P)$, given by

$$(Q)\begin{cases} \min\limits_{\boldsymbol{y}} & \boldsymbol{y}^\top \boldsymbol{b} \\ s.t. & \boldsymbol{y}^\top \mathrm{A} \ge \boldsymbol{c}^\top \\ & \boldsymbol{y} \ge 0. \end{cases}$$

For a perturbed LP $(P(\cdot))$, the corresponding dual perturbed LP is denoted $(Q(\cdot))$. We now define the relevant robustness problems for perturbed LPs.

**Definition 10** (Weak Robustness). A perturbed LP $(P(\cdot))$ is *weakly robust* if there exists $\varepsilon_0 > 0$ such that, for all $\varepsilon \in [0, \varepsilon_0]$, the LP $(P(\varepsilon))$ is feasible and has a bounded value. In other words, $\mathrm{val}\,(P(\varepsilon)) \in \mathbb{R}$, or equivalently, both $(P(\varepsilon))$ and its dual $(Q(\varepsilon))$ are feasible.

**Definition 11** (Strong Robustness). A perturbed LP $(P(\cdot))$ is *strongly robust* if there is a constant solution for both the primal and dual. Formally, there exist two vectors $\boldsymbol{x}^*, \boldsymbol{y}^* \in \mathbb{R}^n$ and a threshold $\varepsilon_0 > 0$ such that $x^*$ is a solution of $(P(\varepsilon))$ and $y^*$ is a solution of $(Q(\varepsilon))$ for all $\varepsilon \in [0, \varepsilon_0]$.

**Definition 12** (Functional Form). Consider a perturbed LP $(P(\cdot))$ that is weakly robust. The *functional form* of this perturbed LP is given by the maps

$$\varepsilon \longmapsto \mathrm{val}\,(P(\varepsilon))$$
$$\varepsilon \longmapsto \boldsymbol{x}^*(\varepsilon),$$

where $\boldsymbol{x}^*(\varepsilon)$ is an optimal vector for $(P(\varepsilon))$, for all $\varepsilon \in (0, \varepsilon_0]$, and for some threshold $\varepsilon_0 > 0$.

Strong robustness can be interpreted as a property of an optimal basis as follows. A perturbed LP is strongly robust if there is an optimal basis, that is, a subset of inequalities that uniquely determine a solution, of the unperturbed primal and dual LP that remains optimal upon small positive perturbations. On top of this robustness of the basis, the corresponding optimal solutions defined by the basis, for the primal and dual LPs, do not depend on the parameter $\varepsilon$ of the perturbation.

Note that the marginal value of Mills [33] depends only on the first order of the perturbations, whereas weak robustness, strong robustness, and the functional form problem depend on all higher terms.

### 5.2. Value Positivity and Robustness Problems

Before describing the connection between robustness for perturbed LPs and value positivity for polynomial matrix games, we formalize the assumption we consider over the perturbed LPs.

**Definition 13** (A Priori Bound). A perturbed LP $(P(\cdot))$ has an *a priori bound* if there exists $\beta > 0$ and $\varepsilon_0 > 0$ such that, for all $\varepsilon \in [0, \varepsilon_0]$, if $\boldsymbol{x}_\varepsilon$ and $\boldsymbol{y}_\varepsilon$ are feasible for $(P(\varepsilon))$ and $(Q(\varepsilon))$, respectively, then $\boldsymbol{x}_\varepsilon^\top \mathbf{1} + \boldsymbol{y}_\varepsilon^\top \mathbf{1} \le \beta$. An a priori bound is said to have polynomial binary size if both $\beta$ and $\varepsilon_0$ have polynomial binary size.

Note that an a priori bound is not concerned with the feasibility of the perturbed LP. Indeed, a perturbed LP whose primal and dual LPs are both infeasible also has an a priori bound by vacuity. Nonetheless, if a perturbed LP $(P(\cdot))$ has an a priori bound and $(P(0))$ is feasible, then the perturbed LP is weakly robust. Indeed, if $(P(0))$ is feasible and bounded, then so is $(Q(0))$. By Martin [30, theorem 1.1], the value function val $(P(\cdot))$ is continuous at zero. In particular, $(P(\varepsilon))$ is feasible for $\varepsilon$ small enough; that is, it is weakly robust.

The existence of an a priori bound is derived from the unperturbed LP as follows:

**Lemma 11** (Sufficient Conditions for an A Priori Bound). *Consider a perturbed LP $(P(\cdot))$. If the unperturbed LP $(P(0))$ and its dual $(Q(0))$ have nonempty and bounded feasible regions, then $(P(\cdot))$ has an a priori bound of polynomial binary size, which can be computed in polynomial time.*

Lemma 11 is closely related to Martin [30, lemma 3.1], which states that if the unperturbed LP has a nonempty and bounded region, then, for each possible continuous perturbation (not necessarily polynomial), the corresponding perturbed LP has a uniformly bounded feasible region for all $\varepsilon$ small enough. The main difference is that because we focus on polynomial perturbations, we prove that the a priori bound has polynomial binary size and a polynomial-time procedure to compute it.

**Proof of Lemma 11.** Consider a perturbed LP $(P(\cdot))$ of size $m \times m$, degree $K$, and integer entries of size bounded by $B$ such that $(P(0))$ and its dual $(Q(0))$ have nonempty and bounded feasible regions. We closely follow the proof of Martin [30, lemma 3.1] while keeping track of explicit numerical bounds.

Consider the LP with the same feasible region as $(P(0))$ but different objective given by

$$\begin{cases} \max\limits_{\boldsymbol{x}} & \mathbf{1}^\top \boldsymbol{x} \\ s.t. & \mathrm{A}(0)\boldsymbol{x} \le \boldsymbol{b}(0) \\ & \boldsymbol{x} \ge 0. \end{cases}$$

Because the feasible region of $(P(0))$ is bounded, it has a finite value. By duality, its dual LP, that is,

$$\begin{cases} \min\limits_{\boldsymbol{y}} & \boldsymbol{y}^\top \boldsymbol{b}(0) \\ s.t. & \boldsymbol{y}^\top \mathrm{A}(0) \ge \mathbf{1}^\top \\ & \boldsymbol{y} \ge 0, \end{cases}$$

is feasible. In particular, it has a basic solution $y_0$, which has polynomial binary size and can be computed in polynomial time. Indeed, using Cramer's rule on a basic solution, we deduce that the coordinates of $\boldsymbol{y}_0$ have size at most the determinant of a matrix involving entries in A(0) and 1. Therefore, because all coefficients of A(0) are bounded by $B$, the size of each coordinate of $\boldsymbol{y}_0$ is bounded by $m!B^m \le (Bm)^m$, so their binary size is at most $m\mathrm{bit}\,(Bm)$.

Define $\beta_1 := \boldsymbol{y}_0^\top \boldsymbol{b}(0) + 1$ and $\varepsilon_1 := (2(Bm)^{m+1}K)^{-1}$. We claim that for all $\varepsilon \in [0, \varepsilon_1]$, every feasible vector $\boldsymbol{x}_\varepsilon$ of $(P(\varepsilon))$ satisfies that $\boldsymbol{x}_\varepsilon^\top 1 \le 2\beta_1$. Indeed, first note that because $\varepsilon_1$ is sufficiently small, we have that for all $\varepsilon \in [0, \varepsilon_1]$,

$$\boldsymbol{y}_0^\top \mathrm{A}(\varepsilon) \ge \frac{1}{2}\mathbf{1}^\top \quad \text{and} \quad \boldsymbol{y}_0^\top \boldsymbol{b}(\varepsilon) \le \beta_1.$$

This is because

$$\boldsymbol{y}_0^\top \mathrm{A}(\varepsilon) \geq \boldsymbol{y}_0^\top \mathrm{A}(0) - \boldsymbol{y}_0^\top 1 B K \varepsilon_1 \geq 1 - (Bm)^{m+1} K \varepsilon_1 \geq \frac{1}{2}$$

and also

$$\boldsymbol{y}_0^\top \boldsymbol{b}(\varepsilon) \leq \boldsymbol{y}_0^\top \boldsymbol{b}(0) + \boldsymbol{y}_0^\top 1 B K \varepsilon_1 \leq \boldsymbol{y}_0^\top \boldsymbol{b}(0) + (Bm)^{m+1} K \varepsilon_1 \leq \boldsymbol{y}_0^\top \boldsymbol{b}(0) + 1 = \beta_1.$$

Then, consider a feasible vector $\boldsymbol{x}_\varepsilon$ of $(P(\varepsilon))$. Because $\mathrm{A}(\varepsilon)\boldsymbol{x}_\varepsilon \leq \boldsymbol{b}(\varepsilon)$ and $\boldsymbol{x}_\varepsilon \geq 0$, we have that

$$\boldsymbol{x}_\varepsilon^\top 1 = 1^\top \boldsymbol{x}_\varepsilon \leq 2\boldsymbol{y}_0^\top \mathrm{A}(\varepsilon)\boldsymbol{x}_\varepsilon \leq 2\boldsymbol{y}_0^\top \boldsymbol{b}(\varepsilon) \leq 2\beta_1.$$

Using a similar construction with the dual, we obtain another pair, $(\beta_2, \varepsilon_2)$. Then, the pair $\beta := 2\beta_1 + 2\beta_2$, and $\varepsilon_0 := \min(\varepsilon_1, \varepsilon_2)$ is an a priori bound for $(P(\cdot))$ of polynomial binary size. Because we only require to solve two LPs for its construction, we can compute it in polynomial time. □

We now describe the connection between robustness for perturbed LPs and value positivity for polynomial matrix games.

**Theorem 5.** *There is a polynomial-time reduction from weak (respectively, strong) robustness and the functional form of perturbed LPs with an a priori bound to value positivity (respectively, uniform value positivity) and the functional form of polynomial matrix games.*

On the one hand, the strong robustness of an LP implies the existence of a vector $\boldsymbol{x}^*$ that is optimal for all positive small parameters. On the other hand, uniform value positivity of polynomial matrix games implies the existence of a strategy $p_0$ that ensures a positive value for all small parameters, but $p_0$ does not need to be optimal for every small parameter. Therefore, at least intuitively, it is not immediately clear that strong robustness of perturbed LPs reduces to uniform value positivity of polynomial matrix games. Theorem 5 shows that this is indeed the case.

Note that robustness problems of perturbed LPs with an a priori bound can be solved by computing the value of unperturbed LPs. Indeed, by Theorem 5, robustness problems of perturbed LPs reduce to value-positivity problems of polynomial matrix games. In turn, value-positivity problems are solved by computing the value of various unperturbed matrix games. Because computing the value of matrix games consists of solving an unperturbed LP, we conclude that robustness problems of perturbed LPs can be solved by computing the value of unperturbed LPs.

**Reduction.** The reduction is based on Von Stengel [50], a simplification of Adler's work [1] to fill the gap left by Dantzig [14]. The reduction of Von Stengel [50] presents a single matrix game with entries computed from the matrix and vectors of the LP being reduced. The matrix game is an extension of Dantzig's matrix game [14]. We accommodate this reduction in the context of polynomial coefficients instead of fixed coefficients.

**Main Idea.** A key insight the so-called *Karp-reduction* of Von Stengel [50], also present in Adler [1], is the introduction of slack variables and bounds. For unperturbed LPs, these bounds are computable in strongly polynomial time when coefficients are algebraic; see Adler [1]. Our assumption of an a priori bound for perturbed LPs is crucial to extend the reduction to the perturbed case. A detailed construction is given in Section 5.3.

## 5.3. Detailed Proofs

In this section, we prove Theorem 5. We start by recalling how solving an LP can be reduced to solving a matrix game according to Von Stengel [50].

**5.3.1. Previous Results.** A convenient restatement of the reduction from LPs to matrix games given in Von Stengel [50, theorem 7, p. 18] is the following:

**Lemma 12** (LP Reduction to Matrix Games). *Consider a primal LP (P), its dual LP (Q), and a bound $\beta$ such that, if $x$ is feasible in (P) and $y$ is feasible in (Q), then $\boldsymbol{x}^\top \mathbf{1} + \boldsymbol{y}^\top \mathbf{1} \leq \beta$. The matrix game*

$$\mathrm{M} := \begin{pmatrix} 0 & \mathrm{A} & -\boldsymbol{b} & -1 \\ -\mathrm{A}^\top & 0 & \boldsymbol{c} & -1 \\ \boldsymbol{b}^\top & -\boldsymbol{c}^\top & 0 & \beta \end{pmatrix}$$

*satisfies the following properties:*

1. *The value of the game is less or equal to zero; that is,* val $(M) \leq 0$.
2. *We have that* val $(M) = 0$ *if and only if the LPs (P) and (Q) are feasible.*
3. *If* val $(M) = 0$ *with optimal strategies* $\boldsymbol{p}^* = (\boldsymbol{y}^*, \boldsymbol{x}^*, t^*)$ *and* $\boldsymbol{q}^* = (\boldsymbol{y}^*, \boldsymbol{x}^*, t^*, 0)$, *then the LPs (P) and (Q) have optimal solutions* $\boldsymbol{x}^*/\boldsymbol{t}^*$ *and* $\boldsymbol{y}^*/\boldsymbol{t}^*$, *respectively.*

**Remark 1.** Some differences between Lemma 12 and Von Stengel [50, theorem 7, p. 18] are the following: (i) the payoff matrix M corresponds to the payoff matrix of the column-player in Von Stengel [50], and (ii) we omit the conclusions of the case val (M) $< 0$ because we will not use them.

**5.3.2. Technical Proofs.** We now turn to Proof of Theorem 5.

**Proof of Theorem 5.** We proceed in four steps: first, we introduce a useful polynomial matrix game; second, we relate weak robustness to value positivity; third, we relate strong robustness to uniform value positivity; and fourth, we relate the functional forms.

Step 1. Introduction of a polynomial matrix game.

Fix a perturbed LP $(P(\cdot))$ of degree $K$ given by

$$(P(\varepsilon)) \begin{cases} \max\limits_{\boldsymbol{x}} & \boldsymbol{c}(\varepsilon)^\top \boldsymbol{x} \\ s.t. & \mathrm{A}(\varepsilon)\boldsymbol{x} \leq \boldsymbol{b}(\varepsilon) \\ & \boldsymbol{x} \geq 0. \end{cases}$$

By assumption, $(P(\cdot))$ has an a priori bound $\beta$, which bounds all primal and dual feasible points in an interval $[0, \varepsilon_0]$. For all $\varepsilon \in [0, \varepsilon_0]$, consider the following polynomial matrix game:

$$\mathrm{M}(\varepsilon) := \begin{pmatrix} 0 & \mathrm{A}(\varepsilon) & -\boldsymbol{b}(\varepsilon) & -1 \\ -\mathrm{A}^\top(\varepsilon) & 0 & \boldsymbol{c}(\varepsilon) & -1 \\ \boldsymbol{b}^\top(\varepsilon) & -\boldsymbol{c}^\top(\varepsilon) & 0 & \beta \end{pmatrix}.$$

Because $\beta$ is an a priori bound, Lemma 12 applies so that for all $\varepsilon \in [0, \varepsilon_0]$, the following properties hold.

1. The value of the matrix game $\mathrm{M}(\varepsilon)$ is less or equal to zero; that is, val $(\mathrm{M}(\varepsilon)) \leq 0$.
2. Moreover, val $(\mathrm{M}(\varepsilon)) = 0$ if and only if the LPs $(P(\varepsilon))$ and $(Q(\varepsilon))$ are feasible.
3. If val $(\mathrm{M}(\varepsilon)) = 0$ with optimal strategies $\boldsymbol{p}_\varepsilon^{*\top} = (\boldsymbol{y}_\varepsilon^*, \boldsymbol{x}_\varepsilon^*, t_\varepsilon^*)$ and $\boldsymbol{q}_\varepsilon^* = (\boldsymbol{y}_\varepsilon^*, \boldsymbol{x}_\varepsilon^*, t_\varepsilon^*, 0)$, then the LPs $(P(\varepsilon))$ and $(Q(\varepsilon))$ have optimal solutions $\boldsymbol{x}_\varepsilon^*/t_\varepsilon^*$ and $\boldsymbol{y}_\varepsilon^*/t_\varepsilon^*$, respectively.

Note that val $(\mathrm{M}(0)) = 0$ because the LP $(P(0))$ is feasible and bounded by assumption.

Step 2. Weak robustness to value positivity.

We claim that the perturbed LP $(P(\cdot))$ is weakly robust if and only if the polynomial matrix game $\mathrm{M}(\cdot)$ is value positive. Indeed, if $\mathrm{M}(\cdot)$ is value positive, then there exists $\varepsilon_0 > 0$ such that, for all $\varepsilon \in [0, \varepsilon_0]$, we have that val $\mathrm{M}(\varepsilon) \geq 0$. Because val $\mathrm{M}(\varepsilon) \leq 0$ by construction, for all $\varepsilon \in [0, \varepsilon_0]$, we have that val $\mathrm{M}(\varepsilon) = 0$. Moreover, by Lemma 12, we conclude that $(P(\varepsilon))$ and $(Q(\varepsilon))$ are feasible; that is, $(P(\cdot))$ is weakly robust.

Conversely, if $\mathrm{M}(\cdot)$ is not value positive, then, for all $\varepsilon_0 > 0$, there exists $\varepsilon \in [0, \varepsilon_0]$ such that val $\mathrm{M}(\varepsilon) < 0$. By Lemma 12, this implies that one of $(P(\varepsilon))$ or $(Q(\varepsilon))$ is not feasible. In other words, $(P(\cdot))$ is not weakly robust.

Step 3. Strong robustness to uniform value positivity.

We now claim that the perturbed LP $(P(\cdot))$ is strongly robust if and only if the polynomial matrix game $\mathrm{M}(\cdot)$ is uniform value positive. Indeed, if $\mathrm{M}(\cdot)$ is uniform value positive, then, there exists a fixed strategy $\boldsymbol{p}^* = (\boldsymbol{x}^*, \boldsymbol{y}^*, t^*)$ and another threshold $\varepsilon_1 > 0$ such that $\boldsymbol{p}^{*\top}\mathrm{M}(\varepsilon) \geq 0$ for all $\varepsilon \in [0, \varepsilon_1]$. Because val $\mathrm{M}(\varepsilon) \leq 0$ for all $\varepsilon \in [0, \varepsilon_0]$, then val $\mathrm{M}(\varepsilon) = 0$ and the strategy $\boldsymbol{p}^*$ is optimal for $\mathrm{M}(\varepsilon)$ for all $\varepsilon \in [0, \min(\varepsilon_0, \varepsilon_1)]$. By Lemma 12, $p^*$ encodes a fixed optimal solution $\boldsymbol{x}^*/t^*$ for $(P(\varepsilon))$ and also a fixed optimal solution $\boldsymbol{y}^*/t^*$ for $(Q(\varepsilon))$, for all $\varepsilon \in [0, \min(\varepsilon_0, \varepsilon_1)]$, so $(P(\cdot))$ is strongly robust.

Conversely, assume that $(P(\cdot))$ is strongly robust. We will show that $\mathrm{M}(\cdot)$ is uniform value positive. Because $(P(\cdot))$ is strongly robust, consider the corresponding (fixed) optimal solutions $\boldsymbol{x}^*$ and $\boldsymbol{y}^*$ of $(P(\varepsilon))$ and $(Q(\varepsilon))$, respectively, for all $\varepsilon \in [0, \varepsilon_0]$. Then, define $t := (\sum_i \boldsymbol{x}_i^* + \sum_j \boldsymbol{y}_j^* + 1)^{-1}$, which is a strictly positive scalar, and construct a strategy for the row-player given by

$$\boldsymbol{p}^* := (t\boldsymbol{x}^*, t\boldsymbol{y}^*, t).$$

We claim that $\boldsymbol{p}^*$ is a witness of uniform value positivity for $\mathrm{M}(\cdot)$. Indeed, fix $\varepsilon \in [0, \varepsilon_0]$. We will show that the strategy $\boldsymbol{p}^*$ guarantees a payoff greater or equal to zero in the matrix game $\mathrm{M}(\varepsilon)$.

1. Because $\boldsymbol{x}^*$ is feasible in $(P(\varepsilon))$, we have that $\mathrm{A}(\varepsilon)\boldsymbol{x}^* \le \boldsymbol{b}(\varepsilon)$. Therefore, $-(t\boldsymbol{x}^*)^\top \mathrm{A}^\top(\varepsilon) + t\boldsymbol{b}^\top(\varepsilon) \ge 0$.
2. Because $\boldsymbol{y}^*$ is feasible in $(Q(\varepsilon))$, we have that $(y^*)^\top \mathrm{A}(\varepsilon) \ge c^\top(\varepsilon)$. Therefore, $(t\boldsymbol{y}^*)^\top \mathrm{A}(\varepsilon) - t\boldsymbol{c}^\top(\varepsilon) \ge 0$.
3. Because $\boldsymbol{x}^*$ and $\boldsymbol{y}^*$ are optimal, by strong duality, $\boldsymbol{c}^\top(\varepsilon)\boldsymbol{x}^* = (\boldsymbol{y}^*)^\top \boldsymbol{b}(\varepsilon)$. In particular, $-(t\boldsymbol{y}^*)^\top \boldsymbol{b}(\varepsilon) + (t\boldsymbol{x}^*)^\top \boldsymbol{c}(\varepsilon) \ge 0$.
4. Because $\boldsymbol{x}^*$ and $\boldsymbol{y}^*$ are feasible and the perturbed LP has an a priori bound of $\beta$, we have that $(\boldsymbol{x}^*)^\top \mathbf{1} + (\boldsymbol{y}^*)^\top \mathbf{1} \le \beta$. Therefore, because $t > 0$, one has that $-(t\boldsymbol{x}^*)^\top \mathbf{1} - (t\boldsymbol{y}^*)^\top \mathbf{1} + t\beta \ge 0$.
5. Because $\boldsymbol{x}^*$ and $\boldsymbol{y}^*$ are feasible, and by the choice of $t$, we have that $p^*$ is a probability measure.

In conclusion, $\boldsymbol{p}^*$ is a fixed strategy that guarantees a payoff greater or equal to zero in the matrix game $\mathrm{M}(\varepsilon)$. This concludes the proof that strong robustness of perturbed LPs reduces to uniform value positivity of polynomial matrix games.

Step 4. Functional form to functional form.

Assume $(P(\cdot))$ is weakly robust. Then, the functional form of the polynomial matrix game $\mathrm{M}(\cdot)$ encodes the functional form of $(P(\cdot))$. Indeed, because the perturbed LP is weakly robust, there exists $\varepsilon_0 > 0$ such that, for all $\varepsilon \in [0, \varepsilon_0]$, we have that $\mathrm{val}\,(\mathrm{M}(\varepsilon)) = 0$. Therefore, every pair of optimal strategies $\boldsymbol{p}_\varepsilon^{*\top} = (\boldsymbol{y}_\varepsilon^*, \boldsymbol{x}_\varepsilon^*, t_\varepsilon^*)$ and $\boldsymbol{q}_\varepsilon^* = (\boldsymbol{y}_\varepsilon^*, \boldsymbol{x}_\varepsilon^*, t_\varepsilon^*, 0)$ encode optimal solutions $\boldsymbol{x}_\varepsilon^*/t_\varepsilon^*$ and $\boldsymbol{y}_\varepsilon^*/t_\varepsilon^*$ for $(P(\varepsilon))$ and $(Q(\varepsilon))$, respectively. Lastly, the functional form of the value of $(P(\cdot))$ is given by

$$\mathrm{val}\,(P(\varepsilon)) = \boldsymbol{c}(\varepsilon)^\top \left(\frac{\boldsymbol{x}_\varepsilon^*}{t_\varepsilon^*}\right).$$

Therefore, it is enough to solve the functional form problem for polynomial matrix games to solve the functional form problem for perturbed LPs. □

## 6. Value Positivity and Stochastic Games

In this section, we exploit the connection between stochastic games (Shapley [41]) and polynomial matrix games developed in (Attia and Oliu-Barton [3–5], Oliu-Barton [36]). We connect value positivity with lower bounds on the discounted and undiscounted values of a stochastic game and uniform value positivity with the existence of an optimal stationary strategy in undiscounted stochastic games.

Stochastic games (Shapley [41]) extend matrix games to a dynamic interaction between two opponents. They are played by stages and described by a finite set of states, finite sets of actions, a reward function, and a transition function. At each stage, knowing the current state, both players simultaneously choose an action; this choice determines a stage reward and the probability distribution for the next state. The notion of value in stochastic games depends on how the stage rewards are aggregated. In the discounted value, they are weighted decreasingly at a constant rate. The undiscounted value corresponds to the limit of the discounted values as the discount rate vanishes (Bewley and Kohlberg [9], Mertens and Neyman [32]).

The characterization and computation of the discounted and undiscounted values of a stochastic game is a central problem in game theory (Solan and Vieille [43]). In a series of papers (Attia and Oliu-Barton [3–5], Oliu-Barton [36]), a new approach was proposed, where the key ingredient is a *family of polynomial matrix games*, denoted $\{M[z] : z \in \mathbb{R}\}$. For each $z \in \mathbb{R}$ and each $\varepsilon \in (0,1]$, the rows (respectively, columns) of the matrix $M[z](\varepsilon)$ correspond to a *pure stationary* strategy of the row-player (respectively, column-player) in the stochastic game, that is, an action choice for each state. For a stochastic game with $n$ states and $m$ actions per state, these matrices are of size $m^n$. The main connection between the parameterized polynomial matrix games and stochastic games is the following.

**Theorem 6** (Attia and Oliu-Barton [3, Theorem 1]). *Consider a stochastic game, and let $M[z]$ be the corresponding polynomial matrix game for a fixed $z \in \mathbb{R}$. Then, for all $\varepsilon \ge 0$, we have that* $\mathrm{val}\, M[z](\varepsilon) \ge 0$ *if and only if the $\varepsilon$-discounted value of the stochastic game is at least $z$.*

Applied to this family of polynomial matrix games, value positivity and uniform value positivity provide useful insights for stochastic games.

**Lemma 13** (Stochastic Games and Value Positivity). *Consider a stochastic game, and let $M[z]$ be the corresponding polynomial matrix game for a fixed $z \in \mathbb{R}$. Then, $M[z]$ is value positive if and only if the discounted value of the stochastic game is at least $z$ for all sufficiently small discount rates.*

**Proof.** This follows directly from Theorem 6. □

**Lemma 14** (Stochastic Games and Uniform Value Positivity). *Consider a stochastic game, and let $M[z]$ be the corresponding polynomial matrix game for a fixed $z \in \mathbb{R}$. If there exists a fixed stationary strategy that guarantees $z$ in all discounted stochastic games with sufficiently small discount rates, then $M[z]$ is uniform value positive.*

**Proof.** Again, this follows from Theorem 6 and the definition of uniform value positivity of polynomial matrix games. □

### Limitations

The reverse implication of Lemma 14 is not known to be true. This is the case, as the connection between $\mathrm{M}[z]$ and the stochastic game is via their values and a restricted set of strategies (Attia and Oliu-Barton [4]).

### Blackwell Optimality

It is worth noting that by instantiating $z$ with the undiscounted value $v$ of the stochastic game, Lemma 14 connects the uniform value positivity of $M[v]$ with a weak version of Blackwell optimality (which we recall is the existence of a fixed stationary strategy that is optimal in all $\varepsilon$-discounted stochastic games for $\varepsilon$ sufficiently small). Indeed, if $p$ is a stationary strategy that guarantees $v$ in all $\varepsilon$-discounted stochastic games for $\varepsilon$ small enough, then $\mathrm{M}[v]$ is uniform value positivity. Note that, in this case, $p$ is an optimal stationary strategy in the undiscounted stochastic game, as it guarantees $v$, but not necessarily Blackwell optimal, as the $\varepsilon$-discounted might be larger than $v$.

We now illustrate Lemmas 13 and 14 with an example.

**Example 5** (Big Match (Blackwell and Ferguson [10])). Consider the following stochastic game, popularized by Blackwell and Ferguson [10]:

$$\begin{pmatrix} 1^* & 0^* \\ 0 & 1 \end{pmatrix}.$$

In this game, the "$*$" indicates an absorbing payoff. In other words, when the top row is played, the corresponding stage payoff is fixed for all future stages, and when the bottom row is played, the stage payoff is recorded once, and the game starts over again. Its value, both in the discounted and undiscounted cases, is $1/2$.

### Value Positivity Analysis

By analyzing the value positivity and uniform value positivity of the corresponding polynomial matrix games, we recover the following properties of this game: (i) for all sufficiently small discount rates, the discounted value is $1/2$; and (ii) whereas the row-player has a unique optimal stationary strategy in every $\varepsilon$-discounted game, there is no fixed optimal stationary strategy in the undiscounted game. To see this, fix $z = 1/2$. Then, up to a positive constant,

$$\mathrm{M}[z](\varepsilon) = \begin{pmatrix} 1 & -1 \\ -\varepsilon & \varepsilon \end{pmatrix}.$$

Note that $\mathrm{M}[z]$ is value positive but not uniform value positive. Indeed, for all $\varepsilon \in [0, 1]$, we have $\operatorname{val} \mathrm{M}[z](\varepsilon) = 0$, and the unique optimal strategy of the row-player is $\boldsymbol{p}^*(\varepsilon) = (\varepsilon/(1+\varepsilon), 1/(1+\varepsilon))^\top$, whereas for each fixed stationary strategy $\boldsymbol{p} \in \Delta[m]$ of the row-player (i.e., not depending on $\varepsilon$), there exists a stationary strategy $q$ of the column-player that leads to a strictly negative payoff as $\varepsilon$ goes to zero.

By Lemma 13, the discounted value of the Big Match is at least $z$ for all sufficiently small discount rates, and by Lemma 14, there is no fixed stationary strategy for the row-player that guarantees $z$ in all discounted games with a sufficiently small discount rate. Similarly, by transposing our results to the column-player, we obtain the existence of a fixed strategy of the column-player that guarantees at most $z$ for all sufficiently small discount rates, implying that the discounted value is at most $z$ in these games. We have thus shown the desired properties (i) and (ii).

### Comments

We finish this section with a few comments on the applicability of value positivity and uniform value positivity to stochastic games.

- *Size of the polynomial matrix games.* By construction, the polynomial matrix games are of size $m^n$ and hence exponential and thus less tractable. A notable exception is absorbing games, a well-studied class of stochastic games (which includes the Big Match) where the state changes at most once during the game. In this class, the corresponding polynomial matrices are of size $m$.
- *Complexity of the undiscounted value*. Lemmas 13 and 14 are particularly relevant when replacing $z$ with the undiscounted value of the stochastic game. However, the undiscounted value is an algebraic number whose degree can be as high as $m^n$ (Oliu-Barton [36, proposition 1]). Because our algorithms require rational entries in the

polynomial matrix game, they may not be directly applicable in this case. A notable exception is the class of stochastic games where at most one player controls the transition in each state. In this class, which includes turned-based stochastic games and Markov decision processes, the discounted and undiscounted values are rational expressions of the data.

- *Stationary $\varepsilon$-optimal strategies.* Whereas discounted stochastic games admit optimal stationary strategies (Shapley [41]), undiscounted games generally do not admit $\varepsilon$-optimal stationary strategies. Some notable exceptions are the following. First, ergodic games, where every pair of stationary strategies induces an ergodic Markov chain over the states (Hoffman and Karp [22]), admit optimal stationary strategies. Second, recursive or terminal reward games (Everett [17]), where the outcome of a single turn is either a real number (the terminal reward) or another state of the game (but not both), and reachability or safety games (De Alfaro et al. [16]), where the objective of one player is to reach a set of states and the opponent wants to prevent it, admit $\varepsilon$-optimal stationary strategies. Therefore, the notion of uniform value positivity is particularly relevant in these classes of stochastic games.

## 7. Extensions

In this section, we discuss a couple of natural extensions of our results.

### 7.1. Nonpolynomial Perturbations

We explain the extent to which our algorithms extend to nonpolynomial perturbations of a matrix game.

**Value Positivity.** Algorithm 1 requires a lower bound on some right neighborhood of zero where the sign of val $\mathrm{M}(\cdot)$ remains constant. Lemma 4 provides a lower bound for polynomial perturbations. In contrast, general analytical functions do not allow such a lower bound. For example, consider the function $x \longmapsto x\sin(1/x)$, which has infinitely many roots arbitrarily close to zero. Therefore, to extend our algorithm, we need to restrict the family of perturbations allowed and ensure a strictly positive lower bound on root separation.

**Functional Form.** Algorithm 2 requires a bound on some right neighborhood of zero where the functional form of val $\mathrm{M}(\cdot)$ and $p^*(\cdot)$ belongs to a particular family of functions. In the case of polynomial perturbations, it is the family of rational functions. Moreover, this family of functions must allow an efficient identification procedure, that is, recovering a particular function of this family with little information. In the case of rational functions, because the degree is bounded, we can identify a function by sufficiently many evaluations.

**Uniform Value Positivity.** Algorithm 3 requires a finite expansion of the perturbation to terminate. Moreover, the functions in the expansion need to be ordered in the following sense. Denote the general perturbed matrix game

$$\mathrm{M}(\varepsilon) = \sum_{k=0}^{K} \mathrm{M}_k f_k(\varepsilon).$$

Then, it must hold that $\lim_{\varepsilon \to 0^+} f_k(\varepsilon)/f_{k-1}(\varepsilon) = 0$ for all $k \in [K]$. Under this condition, Algorithm 3 can be extended to more general perturbations with no changes.

### 7.2. Constrained Matrix Games

Players may be constrained in the mixed strategies they are allowed to play. A classical restriction over players' strategies is that for each player, their mixed strategy must belong to a polytope strictly included in the set of all mixed strategies; that is, they must satisfy finitely many linear inequalities. We can incorporate these restrictions in the computation of the value of a matrix game. Indeed, there are two changes one must make to the LP that computes the value. First, the constraints over the row-player's strategies must be incorporated as new constraints on the feasible strategies $p$. Second, instead of having an inequality for each action of the column-player, we must have an inequality for each vertex of the polytope of feasible strategies for the column-player. To maintain the polynomial-time complexity of solving the LP, the polytope must have a polynomial number of vertices, and its inequalities must be given by rational coefficients of polynomial binary size. Under these conditions, our results extend to polynomial matrix games where strategies satisfy linear constraints.

### 7.3. Perturbed Stochastic Games

Consider a *polynomial stochastic game*, that is, where the payoff and transition functions are polynomials in some parameter $\delta \in \mathbb{R}$. This model extends linearly perturbed stochastic games, studied in Attia et al. [6]. We are interested in small, positive parameters and focus on the discounted case. For each discount rate $\varepsilon \in (0,1]$ and each

$\delta \geq 0$, denote the corresponding discounted value by $v(\varepsilon, \delta)$. As explained in Section 6, each stochastic game has a corresponding family of polynomial matrix games. Consider the family $\{M[z, \varepsilon] : (z, \varepsilon) \in \mathbb{R} \times (0,1]\}$ of polynomial matrix games with parameter $\delta$. The following statements are similar to Lemmas 13 and 14 but fix the discount rate $\varepsilon$ and reformulated in terms of the parameter $\delta$.

**Lemma 15.** *Consider a polynomial stochastic game, and let $M[z, \varepsilon]$ be the corresponding polynomial matrix game for some fixed parameter $z \in \mathbb{R}$ and discount rate $\varepsilon \in (0,1]$. Then, $M[z, \varepsilon]$ is value positive if and only if $v(\varepsilon, \delta) \geq z$ for all sufficiently small $\delta$.*

**Lemma 16.** *Consider a stochastic game, and let $M[z, \varepsilon]$ be the corresponding polynomial matrix game for a fixed parameter $z \in \mathbb{R}$ and discount rate $\varepsilon \in (0,1]$. If there exists a fixed stationary strategy that guarantees $z$ in all $\varepsilon$-discounted polynomial stochastic games with sufficiently small $\delta$, then $M[z, \varepsilon]$ is uniform value positive.*

## 8. Conclusion

We have studied the stability of matrix games; presented new algorithmic problems called value positivity, uniform value positivity, and functional form; shown how these problems relate to previous analyses of matrix games; and presented polynomial-time algorithms to solve them. Also, we explained the connection of the value-positivity problems to the robustness of LPs and presented related problems and the corresponding reductions.

### Future Directions

Extending the concept of value positivity to the undiscounted value of polynomial stochastic games is very interesting. In Bourque and Raghavan [11], Nowak and Raghavan [35], Raghavan and Syed [37], Raghavan et al. [38], and Thuijsman and Raghavan [46], it is shown that some classes of stochastic games allow a characterization of the undiscounted value similar to Theorem 2 and Lemma 6. For example, when optimal pure stationary strategies exist, the value of the game corresponds to one of finitely many candidates. Such property would imply algorithms for the value-positivity and functional form problems, whereas the uniform value positivity would require an efficient search in the strategy space.

## References

[1] Adler I (2013) The equivalence of linear programs and zero-sum games. *Internat. J. Game Theory* 42(1):165–177.

[2] Altman E, Feinberg EA, Filar JA, Gaitsgory VA (2001) Perturbed zero-sum games with applications to stochastic and repeated games. Altman E, Pourtallier O, eds. *Advances in Dynamic Games and Applications*, Annals of the International Society of Dynamic Games (Birkhäuser, Boston), 165–181.

[3] Attia L, Oliu-Barton M (2019) A formula for the value of a stochastic game. *Proc. Natl. Acad. Sci. USA* 116(52):26435–26443.

[4] Attia L, Oliu-Barton M (2021) Shapley-snow kernels, multiparameter eigenvalue problems, and stochastic games. *Math. Oper. Res.* 46(3):1181–1202.

[5] Attia L, Oliu-Barton M (2023) Stationary equilibria in discounted stochastic games. *Dynam. Games Appl.* 14:271–284.

[6] Attia L, Oliu-Barton M, Saona R (2024) Marginal values of a stochastic game. *Math. Oper. Res.*, ePub ahead of print March 12, https://doi.org/10.1287/moor.2023.0297.

[7] Avrachenkov KE, Filar JA, Howlett PG (2013) *Analytic Perturbation Theory and Its Applications* (Society for Industrial and Applied Mathematics, Philadelphia).

[8] Basu S, Pollack R, Roy MF (2006) *Algorithms in Real Algebraic Geometry*, Algorithms and Computation in Mathematics, vol. 10 (Springer, Berlin, Heidelberg).

[9] Bewley T, Kohlberg E (1976) The asymptotic theory of stochastic games. *Math. Oper. Res.* 1(3):197–208.

[10] Blackwell D, Ferguson TS (1968) The big match. *Ann. Math. Statist.* 39(1):159–163.

[11] Bourque M, Raghavan TES (2014) Policy improvement for perfect information additive reward and additive transition stochastic games with discounted and average payoffs. *J. Dynam. Games* 1(3):347–361.

[12] Brooks B, Reny PJ (2023) A canonical game—75 years in the making—showing the equivalence of matrix games and linear programming. *Econom. Theory Bull.* 11(2):171–180.

[13] Cauchy A (1832) Calcul des indices des fonctions. *Journal de l'École Polytechnique* 15(25):176–229.

[14] Dantzig G (1951) A proof of the equivalence of the programming problem and the game problem. Koopmans T, ed. *Activity Analysis of Production and Allocation*, Cowles Commission for Research in Economics, vol. 13 (John Wiley and Sons, New York), 330–338.

[15] Dantzig GB (1998) *Linear Programming and Extensions*, vol. 48 (Princeton University Press, Princeton, NJ).

[16] De Alfaro L, Henzinger TA, Kupferman O (2007) Concurrent reachability games. *Theoret. Comput. Sci.* 386(3):188–217.

[17] Everett H (1957) Recursive games. *Contributions to the Theory of Games*, vol. 3 (Princeton University Press, Princeton, NJ), 47–78.

[18] Fiacco AV (1983) *Introduction to Sensitivity and Stability Analysis in Nonlinear Programming* (Academic Press, New York).

[19] Fiacco AV, ed. (2020) *Mathematical Programming with Data Perturbations*, 1st ed. (CRC Press, Boca Raton, FL).

[20] Gal T (1994) *Postoptimal Analyses, Parametric Programming, and Related Topics: Degeneracy, Multicriteria Decision Making, Redundancy* (De Gruyter, Berlin).

[21] Gal T, Greenberg HJ, Hillier FS, eds. (1997) *Advances in Sensitivity Analysis and Parametric Programming*, International Series in Operations Research & Management Science, vol. 6 (Springer, Boston).
[22] Hoffman AJ, Karp RM (1966) On nonterminating stochastic games. *Management Sci.* 12(5):359–370.
[23] Jacobi C (1846) Über die darstellung einer reihe gegebner werthe durch eine gebrochne rationale function. *J. für die reine und angewandte mathematik (Crelles J.)* 1846(30):127–156.
[24] Jeroslow R (1973) Linear programs dependent on a single parameter. *Discrete Math.* 6(2):119–140.
[25] Jeroslow RG (1973) Asymptotic linear programming. *Oper. Res.* 21(5):1128–1141.
[26] Karloff H (1991) *Linear Programming, Progress in Theoretical Computer Science* (Birkhäuser, Boston).
[27] Khachiyan LG (1980) Polynomial algorithms in linear programming. *USSR Computational Math. Math. Phys.* 20(1):53–72.
[28] Kuhn HW, Tucker AW (2014) Nonlinear programming. Giorgi G, Kjeldsen TH, eds. *Traces and Emergence of Nonlinear Programming* (Springer, Basel, Switzerland), 247–258.
[29] Mahler K (1964) An inequality for the discriminant of a polynomial. *Michigan Math. J.* 11(3):257–262.
[30] Martin DH (1975) On the continuity of the maximum in parametric linear programming. *J. Optim. Theory Appl.* 17(3–4):205–210.
[31] Megiddo N (1991) On finding primal- and dual-optimal bases. *ORSA J. Comput.* 3(1):63–65.
[32] Mertens JF, Neyman A (1981) Stochastic games. *Internat. J. Game Theory* 10(2):53–66.
[33] Mills HD (1956) Marginal values of matrix games and linear programs. Kuhn HW, Tucker AW, eds. *Linear Inequalities and Related Systems*, vol. 38 (Princeton University Press, Princeton, NJ), 183–193.
[34] Nilim A, El Ghaoui L (2005) Robust control of Markov decision processes with uncertain transition matrices. *Oper. Res.* 53(5):780–798.
[35] Nowak AS, Raghavan TES (1993) A finite step algorithm via a bimatrix game to a single controller non-zero sum stochastic game. *Math. Prog.* 59(1):249–259.
[36] Oliu-Barton M (2021) New algorithms for solving zero-sum stochastic games. *Math. Oper. Res.* 46(1):255–267.
[37] Raghavan TES, Syed Z (2002) Computing stationary Nash equilibria of undiscounted single-controller stochastic games. *Math. Oper. Res.* 27(2):384–400.
[38] Raghavan TES, Tijs SH, Vrieze OJ (1985) On stochastic games with additive reward and transition structure. *J. Optim. Theory Appl.* 47(4):451–464.
[39] Rockafellar RT (1970) *Convex Analysis*, Princeton Landmarks in Mathematics and Physics (Princeton University Press, Princeton, NJ).
[40] Ruhe G (1991) Parametric flows. *Algorithmic Aspects of Flows in Networks*, Mathematics and Its Applications, vol. 69 (Springer, Dordrecht, Netherlands), 125–153.
[41] Shapley LS (1953) Stochastic games. *Proc. Natl. Acad. Sci. USA* 39(10):1095–1100.
[42] Shapley LS, Snow RN (1950) Basic solutions of discrete games. Kuhn HW, Tucker AW, eds. *Contributions to the Theory of Games (AM-24)*, vol. 1 (Princeton University Press, Princeton, NJ), 27–36.
[43] Solan E, Vieille N (2015) Stochastic games. *Proc. Natl. Acad. Sci. USA* 112(45):13743–13746.
[44] Sorin S (2003) The operator approach to zero-sum stochastic games. Neyman A, Sorin S, eds. *Stochastic Games and Applications*, NATO Science Series, vol. 570 (Springer, Dordrecht, Netherlands), 417–426.
[45] Spielman D, Teng SH (2001) Smoothed analysis of algorithms: Why the simplex algorithm usually takes polynomial time. *Proc. 33rd Annual ACM Sympos. Theory Comput.* (ACM, New York), 296–305.
[46] Thuijsman F, Raghavan TES (1997) Perfect information stochastic games and related classes. *Internat. J. Game Theory* 26(3):403–408.
[47] Tijs SH, Vrieze OJ (1980) Perturbation theory for games in normal form and stochastic games. *J. Optim. Theory Appl.* 30(4):549–567.
[48] Verma T, Kumar A (2020) Matrix games with interval payoffs. *Fuzzy Solution Concepts for Non-cooperative Games*, Studies in Fuzziness and Soft Computing, vol. 383 (Springer, Cham, Switzerland), 1–36.
[49] von Neumann J (1928) Zur theorie der Gesellschaftsspiele. *Math. Ann.* 100(1):295–320.
[50] Von Stengel B (2023) Zero-sum games and linear programming duality. *Math. Oper. Res.* 49(2):1091–1108.